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I. Skrynnyk[1]**,**
D. Dmitrishin[1]**,** DSc, Prof.
A. Stokolos[2], DSc, Prof.,
I.E. Iacob[2]**,** DSc, Prof.
[1] Odessa National Polytechnic University, 1 Shevchenko Ave., Odessa, Ukraine, 65044; e-mail: skrynnyk@opu.ua, dmitrishin@opu.ua
[2] Georgia Southern University, 1332 Southern Drive, Statesboro, GA 30458, USA; e-mail: astokolos@GeorgiaSouthern.edu, ieiacob@GeorgiaSouthern.edu

COMPUTATIONAL ASPECTS OF LARGE-LENGTH CYCLE SEARCH ALGORITHMS FOR NONLINEAR DISCRETE SYSTEMS

"D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable"

Henri Poincaré-Les méthodes nouvelles de
la mécanique céleste, Tome 1, (& 36), 1892.

І.М. Скринник, Д.В. Дмитришин, О.М. Стоколос, І.Е. Якоб. **Обчислювальні аспекти алгоритмів пошуку циклів великих довжин для нелінійних дискретних систем.** Динаміка навіть найпростіших нелінійних дискретних систем є досить складною. Вона включає в себе, як періодичні руху, так і квазіперіодичні або рекурентні. У таких системах майже завжди присутні хаотичні атрактори, природа яких на сьогодні досить добре вивчена, а саме, для широкого класу модельних рівнянь. У багатьох випадках хаотичні атрактори можна моделювати за допомогою періодичних рухів з великими періодами. Пошук таких атракторів і мінімальних інваріантних множин на них є важливим завданням прикладної математики – рішення використовуються в фізичних, хімічних, економічних науках, в теорії кодування, передачі сигналів і ін. Проте математичні результати, засновані на комп'ютерних обчисленнях, вимагають ретельної перевірки на верифікацію, так як самі обчислення проводяться наближено, а хаотичні системи дуже чутливі до похибок обчислень. Один з підходів вирішення завдань пошуку і верифікації циклів заснований на застосуванні методів стабілізації цих циклів. Ці методи можна розділити на дві групи: контроль із запізненням, який використовує знання про попередні стани системи, і прогнозує контроль, який використовує майбутні значення стану системи при відсутності управління. Мета роботи – показати ефективність методу усередненого прогнозуючого контролю пошуку циклів на деяких популярних в технічній літературі динамічних системах. А також сформулювати необхідні умови того, що знайдена орбіта є дійсно циклом.У статті розвиваються методи прогнозуючого контролю: використовується усереднений прогнозуючий контроль, і пропонуються алгоритми пошуку циклів, засновані на властивостях такого контролю. Відзначаються різні особливості роботи алгоритмів в залежності від властивостей вихідної дискретної системи. Запропоновано методи верифікації циклічних точок у вигляді трьох необхідних умов циклічності точки: перевірка малої нев'язки, перевірка періодичності і перевірка локальної асимптотичної стійкості циклу. Для демонстрації роботи алгоритму і чисельного моделювання були обрані відомі двовимірні дискретні системи, такі як Lozi, Henon, Ikeda, Elhadj-Sprott, Multihorseshoe, Prey-Predator. До істотних особливостей цих систем відносяться наявність циклів великих довжин з домінуючим мультиплікатором, тобто в двовимірному випадку з одним великим по модулю мультиплікатором, а другим по модулю меншим одиниці. Для такого класу систем запропонований алгоритм працює особливо ефективно. Розроблений метод можна використовувати і для дослідження залежності топологічних властивостей дискретних динамічних систем від зміни параметрів, вивчення наявності біфуркацій і їх типів.
Ключові слова: нелінійні дискретні системи, стабілізація періодичних рішень, алгоритми пошуку циклів великих довжин

I. Skrynnyk, D. Dmitrishin, A. Stokolos, I.E. Iacob. **Computational aspects of large-length cycle search algorithms for nonlinear discrete systems.** Even the simplest nonlinear discrete systems dynamics is very complex. It includes both periodic movements and quasi-periodic or recurrent ones. In such systems, almost always present are the chaotic attractors, whose nature is currently well studied, at least for a wide class of model equations. In many cases, chaotic attractors can be modeled using periodic motions characterized with large periods. Such attractors' and minimal invariant sets' search represents an important task of applied mathematics, with respect to that the solutions are used in physical, chemical, economic sciences, in coding theory, signal transmission theory and so on. However, mathematical results based on computer calculations require a careful verification, since these calculations themselves are carried out approximately, and the chaotic systems are very sensitive to calculation errors. One of the approaches to solving the cycles search and verification problem is based on the application of these cycles' stabilization methods. These methods can be divided into two groups: delayed control, that uses knowledge on system's previous states, and predictive control, which uses the future values of system state in the absence of control. This study purpose is to demonstrate the

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effectiveness of the cycles search averaged predictive control method on some dynamical systems widely referred to in technical reference sources. Another important goal we aimed onto is to formulate the necessary conditions at which the orbit found actually represents a cycle. The article exposes the elaboration of predictive control methods: the averaged predictive control is used, at that the cycles search algorithms based on such control properties are offered. Noted are various features of algorithms' functioning that depend on the original discrete system properties. Proposed are the cyclic points' verification methods in the form of three necessary conditions of point's cyclicity: checking the smallness of the residual, checking the periodicity and checking the cycle local asymptotic stability. Well-known two-dimensional discrete systems such as Lozi, Henon, Ikeda, Elhadj-Sprott, Multihorseshoe, Prey-Predator have been chosen to demonstrate the algorithm and numerical simulation. These systems' essential features include the presence of large lengths cycles with a dominant multiplier, i.e. when two-dimensional case one multiplier has larger modulus, and another's modulus is less than one. With this class of systems, the proposed algorithm operates particularly efficiently. The developed method can also be used to study the discrete dynamical systems' topological properties dependence on changes in parameters, as well as to study the presence of bifurcations and their types.

Keywords: nonlinear discrete systems, periodic solutions stabilization, search algorithms for large-length cycles

Introduction

The exposed study represents a continuation of the work [1]. Even the simplest nonlinear discrete systems' dynamics is very complex. Such systems are often characterized by extremely unstable motions in phase space, defined as chaotic ones [2]. That is why the study of such systems properties is a very difficult task. Due to its theoretical significance and engineering applicability much attention was paid to this problem in various fields and studies [3, 4]. With the advent of powerful computers, it became possible to study the dynamic systems' properties by numerical methods. New hyperbolic structures of systems, such as strange attractors, can be determined using these methods. However, the possibility to confirm numerical results by rigorous mathematical proofs is limited to only some special cases. Computer proofs require the use of special interval arithmetic analysis [5 – 7], complex by its nature and also applicable not to every case occurring. However, there appear numerous studies in the field of chaotic dynamical systems theory, which are published without a careful verification of the obtained numerical results reliability [8]. And even in the cases of the simplest dynamical systems, even at real possibility to choose a very high accuracy of calculations, we can never say with certainty what we have found: a long cycle, a pseudo cycle, or a strange attractor. Chaotic dynamical systems are very sensitive to initial conditions and to rounding errors in calculations, that is after a few steps the results can vary greatly depending on the chosen calculations accuracy, and long-term prediction is impossible at all, so we are facing the so-called "butterfly effect".

Thus, the development of new methods for finding chaotic or strange attractors and their minimal invariant sets is an important and urgent task of applied mathematics, those solutions being used in physical, chemical, economic sciences, in the coding and signal transmission theories, etc.

Analysis of recent publications and problem statement. It is assumed that the dynamical system has a strange attractor that contains a countable set of unstable cycles at different periods. These cycles define the attractor skeleton, and knowing them we can determine the dynamical system properties.

If using the control action we locally stabilize a cycle, the system trajectory will remain in its neighborhood, i.e., we shall observe regular movements in the system, and the cycle will be known.

To solve the stabilization and search problems, various control schemes have been proposed [9, 10], which can be divided into two large groups: direct and indirect. Indirect methods either use T iteration of the original map or build a system whose order T is several times greater than the original one (T – the desired cycle length). And further one of fixed point search methods is applied. The most common among them is the Newton-Raphson relaxation method and its modifications [11, 12]. Having found all the fixed points, it is necessary to choose from this set of fixed points the periodic ones.

In direct methods all points of a cycle are searched at once, i.e. the cycle as a whole is stabilized. In this case, the original system is closed by control based on the feedback principle: delayed [13 – 16] or predictive [1, 17, 18]. The advantages and disadvantages of such controls are exposed in [19 – 21].

In contrast to interval analysis methods and cycle shadow theory, the cycle search methods based on cycles' stabilization allow us to verify more efficiently our calculations authenticity.

The presented study goal is to show the effectiveness of cycles search averaged predictive control method [1] with some dynamical systems popular in the technical reference sources. Another important goal we aimed onto is to formulate the necessary conditions at which the orbit found actually represents a cycle.

Algorithm mathematical foundation

We consider a nonlinear discrete system, which in the absence of control has the form

$$\hat{x}_{n+1} = f(\hat{x}_n), \ \ x_n \in R^m, \ \ n = 1, \ 2, \ \dots, \tag{1}$$

where $f(x)$ is the differentiable vector function of corresponding dimension. It is assumed that system (1) has an invariant convex set *A*, i.e., if $\xi \in A$, then $f(\xi) \in A$ also will be not necessarily minimal. It is also assumed that there are one or more unstable *T*-cycles $\{\eta_1, \dots, \eta_T\}$ in this system, where all vectors $\eta_1, \dots, \eta_T$ are distinct and belong to an invariant set *A*, i.e. $\eta_{j+1} = f(\eta_j)$, $j = 1, \dots, T-1$ $\eta_1 = f(\eta_T)$. The considered unstable cycles' multipliers are defined as the eigenvalues of Jacobi matrices $\prod_{j=1}^{T} f'(\eta_{T-j+1})$ products having dimensions $m \times m$ at the cycle points. The matrix $\prod_{j=1}^{T} f'(\eta_{T-j+1})$ is called the Jacobi matrix of cycle $\{\eta_1, \dots, \eta_T\}$. As a rule, the system (1) cycles $\{\eta_1, \dots, \eta_T\}$ are not known a priori. Therefore, unknown is the matrix $\prod_{j=1}^{T} f'(\eta_{T-j+1})$ spectrum $\{\mu_1, \dots, \mu_m\}$. The spectrum set elements are called cycle multipliers. Further, we assume that we know some estimate of the cycle multipliers localization set *M*.

Now we shall consider the control system

$$x_{n+1} = F(x_n), \tag{2}$$

where $F(x) = \sum_{j=1}^{N} \vartheta_j f^{((j-1)T+1)}(x)$, $f^{(1)}(x) = f(x)$, $f^{(k)}(x) = f(f^{(k-1)}(x))$ $k = 2, \dots, (N-1)T+1$.

Numbers $\vartheta_1, \dots, \vartheta_N$ are real. It is easy to verify that at $\sum_{j=1}^{N} \vartheta_j = 1$ the system (2) also contains the cycle $\{\eta_1, \dots, \eta_T\}$.

The parameter *N* and coefficients $\vartheta_1, \dots, \vartheta_N$ are chosen so that the system (2) cycle $\{\eta_1, \dots, \eta_T\}$ would be locally asymptotically stable. It is also desirable [23] to fulfill the additional condition: the invariant convex set *A* of system (1) must be invariant for system (2) as well. This requirement will be met, for example, if $0 \le \vartheta_j \le 1$, $j = 1, \dots, N$.

The mathematical basis for the choice of coefficients $\vartheta_1, \dots, \vartheta_N$ is laid by the following statement.

Theorem [1]. Let $f \in C^1$ and the system (1) have an unstable *T*-cycle with multipliers $\{\mu_1, \dots, \mu_m\}$. Then this cycle will be a locally asymptotically stable cycle of system (2) if

$$\mu_j [r(\mu_j)]^T \in D, \ \ j = 1, \dots, m,$$

where $D = \{z \in C : |z| < 1\}$ is the open central unit circle, $r(\mu) = \sum_{j=1}^{N} \vartheta_j \mu^{j-1}$.

To be noted is that instead of system (2), another control system can be considered

$$x_{n+1} = f\left(\vartheta_1 x_n + \sum_{j=2}^{N} \vartheta_j f^{((j-1)T)}(x_n)\right). \tag{3}$$

When $\sum_{j=1}^{N} \vartheta_j = 1$ in system (3) the cycle $\{\eta_1, \dots, \eta_T\}$ is maintained. In addition, this cycle Jacobi matrices for systems (3) and (4) are the same. The advantage of the control system (3) over the system (2) is a smaller number of calculations for the function $f(x)$ values getting.

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Algorithm

If the multipliers of system (1) are known exactly, then $N = m+1$ and coefficients $\vartheta_1, \ldots, \vartheta_{m+1}$ can be chosen from the condition $r(\mu) = \sum_{j=1}^{m+1} \vartheta_j \, \mu^{j-1} = \frac{1}{\prod_{k=1}^{m}(1-\mu_k)} \prod_{k=1}^{m}(\mu - \mu_k)$. Then from the theorem we obtain

Consequence. Let $f \in C^1$ and system (1) have an unstable *T*-cycle with multipliers $\{\mu_1, \ldots, \mu_m\}$, and coefficients $\vartheta_1, \ldots, \vartheta_{m+1}$ are found as above. Then this cycle will be a locally asymptotically stable cycle of systems (2) and (3). Moreover, if the initial point belongs to the cycle basin of attraction, all multipliers of the cycle $\{\eta_1, ..., \eta_T\}$ at systems (2) and (3) are equal to zero, and the convergence to the cycle is superlinear.

So, if we know exactly or approximately the cycle multipliers, then the problem of this cycle local stabilization is solved.

Now we consider the case where the multipliers location on the complex plane is unknown. Using the ideas from the theorem's corollary, we can propose the following scheme of *T*-cycle stabilization. In this case, the coefficients ϑ_j will not necessarily be constants.

a) find the matrix: $f'(x)$,

b) find the vectors: $f^{(s)}(x)$, $s = 1, \ldots, T-1$,

c) now we find the matrix: $f'(f^{(T-1)}(x)) \cdot \ldots \cdot f'(f(x)) \cdot f'(x)$,

d) and find the characteristic polynomial of this matrix: $\sum_{j=1}^{m+1} \vartheta_j(x)\, \mu^{j-1}$,

e) normalize the characteristic polynomial: $\frac{1}{\sum_{j=1}^{m+1} \vartheta_j(x)} \sum_{j=1}^{m+1} \vartheta_j(x)\, \mu^{j-1}$,

f) build a control system:

$$x_{n+1} = F(x_n),$$

where $F(x) = \frac{1}{\sum_{j=1}^{m+1} \vartheta_j(x)} \sum_{j=1}^{m+1} \vartheta_j(x) f^{((j-1)T+1)}(x)$ or

$$F(x) = f\left(\frac{1}{\sum_{j=1}^{m+1} \vartheta_j(x)} \left(\vartheta_1 x + \sum_{j=2}^{m+1} \vartheta_j(x) f^{((j-1)T)}(x) \right) \right).$$

In general, applying the method to the stabilization of chaotic motion characterized with property of mixing, one can expect that after a certain iterations number the trajectory will reach the stabilized cycle basin of attraction. Then the convergence to the cycle will be superlinear.

Note that if instead of ϑ_j we use $|\vartheta_j|$, it is possible to stabilize the cycles of system (1) with multipliers lying in the region $M = D \cup \{\mu : \mathrm{Re}(\mu) \le 0\}$, i.e. with real negative multipliers or lying in the unit circle. At that case the convex invariant sets of system (1) will remain the same for systems (2), (3). In addition, these systems' multipliers corresponding to those multipliers of system (1) that lie in the unit circle become closer to zero.

Now we consider the case when system (1) has one dominant multiplier of *T*-cycle, and the remaining multipliers lie in the central unit circle of the complex plane. Such situations happen quite often, for example, at many popular two-dimensional and three-dimensional mappings [24], i.e. a set of localization multipliers $M = D \cup \{\mu^*\}$, $|\mu^*| > 1$.

If we assume that the dominant multiplier is negative, we can choose the following control scheme:

$$x_{n+1} = f\left(\frac{\vartheta}{1+\vartheta} x_n + \frac{1}{1+\vartheta} f^{(T)}(x_n)\right), \quad (4)$$

where the value ϑ shall be chosen from the condition:

$$\left|\mu^*\left(\frac{\vartheta}{1+\vartheta} + \frac{1}{1+\vartheta}\mu^*\right)^T\right| < 1, \quad (5)$$

from where $\vartheta > 0$.

Let we denote $\mu = -\mu^*$, $\mu > 1$ Suppose that $\mu \le 2^T$. Such a restriction for the cycle multipliers is typical of the cycles appearing as a result of period-doubling bifurcations. In order to satisfy the condition (5), it is necessary and sufficient to fulfill the inequality $\left|\frac{\vartheta}{1+\vartheta} - \frac{1}{1+\vartheta}\mu\right| < \frac{1}{2}$ from where $\frac{1}{3}(-1+2\mu) < \vartheta < 1+2\mu$. Since $\frac{1+2\mu}{\frac{1}{3}(-1+2\mu)} > 3$, then, by choosing $\vartheta = \alpha 3^k$, $k = 1, 2, \ldots$ $0 < \alpha < \frac{1}{3}(-1+2\mu)$, it is easy to make sure that some k we have $\vartheta \in \left(\frac{1}{3}(-1+2\mu), 1+2\mu\right)$ that entails the fulfillment of condition (5).

Thus, by iterating over no more than *T* values of the control parameter ϑ in the system (2) or (3), it is possible to locally stabilize the system *T*-cycle (1). Note that the problem of large multipliers is not the main one for the considered task of cycle stabilization. The main problem refers to the long cycles' small basins of attraction. Therefore, it is necessary either to take a dense grid for initial values or to use rather large iterations number that the point x_n would get to the necessary basin of attraction.

Numerical results verification

Thus, the proposed numerical methods implemented on computers theoretically do solve the problem of determining the dynamical system orbits. However, due to the chaotic dynamics in the original system, following questions remain open: Can we rely on these numerical solutions? How to verify the numerical results? What accuracy of calculations should be chosen?

We shall consider the original system (1) at $m = 2$, and the system with control $x_{n+1} = \frac{\vartheta}{1+\vartheta} f(x_n) + \frac{1}{1+\vartheta} f^{(T+1)}(x_n)$ (system (4) consideration leads to similar results). Let the sequence $\{x_n\}_{n=1}^{\infty}$ be the solution of a system that includes control. We want to control the residual $U_n = \|x_{n+1} - f(x_n)\|$, if the sequence $\{x_n\}$ tends to solve the system (1), then the sequence $\{U_n\}$ tends to zero. Let the control parameter be of order *p*, i.e. $\vartheta \sim 10^p$. For large lengths cycles one can expect that *p* shall be large enough. Then the residual U_n can be estimated as $U_n = \frac{1}{1+\vartheta}\left\| f^{(T+1)}(x_n) - f(x_n)\right\|$. In general case, if the sequence $\{x_n\}$ does not tend to solve system (1), the residual will be of order 10^{-p}, i.e. $U_n \sim 10^{-p}$. To understand that the residual tends to zero necessary is to choose the calculations accuracy p_1, where p_1 there should significantly exceed *p*. Then the first point of control will be the condition

$$U_n \sim 10^{-p_1}, \; n \ge n_1. \quad (6)$$

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The next check point is the verification T periodicity of the obtained numerical solution:

$$\left\| x_{n+T} - x_n \right\| \sim 10^{-p_1}, \; n \ge n_1 . \tag{7}$$

Of course, in addition it is necessary to check that T is the minimum number for which the condition (7) is satisfied.

The third check point is verification of the conditions of theorem 1, i.e.:

$$\left| \mu_j \left(\frac{\vartheta}{1+\vartheta} + \frac{1}{1+\vartheta} \mu_j \right)^T \right| < 1, \; j = 1, 2, \tag{8}$$

where μ_1, μ_2 are the T-cycle multipliers, i.e. the eigenvalues of matrix $\prod_{j=1}^{T} f'(x_{n_2 - j})$ at some $n_2 \ge n_1 + T$.

Conditions (6), (7), (8) are necessary that the found sequence $\{x_n\}$ would be a T-cycle of the system (1). To be noted is that conditions (6) and (7) are not equivalent: the T-cycle of system (2) or (3) may not be the solution of system (1) at all.

The effectiveness of these necessary conditions is due to the fact that they are simple enough to verify.

Examples

Here we illustrate how the averaged predictive control method works for finding large lengths cycles on several model examples of two-dimensional dynamical systems. System (4) was taken as the control system. We succeed to find a significant number of cycles for all considered T; at different initial conditions and different values of parameter ϑ separate cycles are found. Numerical calculations show that with a sufficiently dense grid of initial values, all cycles of a given length can be found. In this case, however, necessary is to ensure that the point x_n remains in the invariant set otherwise it usually escapes to infinity.

The Lozi mapping [25] will be considered in more detail as the simplest from the computational side. Next, numerical simulation results and corresponding graphs for Henon [26], Ikeda [27], Elhadgj-Sprott [28], Multihorseshoe [29] mappings will be presented. For all these mappings, a large number of cycles with a dominant length multiplier up to $T = 1001$ was found. Once again, essential is that the loop length was not the main obstacle to cycle searching much more restrictions were associated with the need to increase the calculations accuracy, that exponentially increased the calculation time. For the Lozi and Henon maps, given are heuristic considerations on the possibility to introduce additional steps in the algorithm which, apparently, can allow finding cycles of Giga and Tera lengths, if sufficiently powerful computers are used for calculations.

The Predator-Prey mapping revealed to be somewhat problematic [30]. Apparently, this is due to the fact that a significant number from among the large lengths cycles have two large multipliers, and the proportion of cycles with a dominant multiplier is not high. However, this hypothesis requires to be confirmed. It was possible to find cycles of lengths only up to $T = 325$.

1. Lozi mapping

$$x_{n+1} = 1 + a\left|x_n\right| + b\, y_n, \; y_{n+1} = x_n, \; a = -1.7, \; b = 0.5 . \tag{9}$$

To find the $T = 1001$ length cycle, we take the value of control parameter at system (4) $\vartheta = 10^{203}$, and determine the calculations accuracy $\delta = 10^{-225}$. We start from the initial point $x_0 = 0.5$, $y_0 = 0$, after a little more than 3000 steps, we determine that the residual value within the specified accuracy is zero, i.e. $U_n \sim \delta$ at $n > 3500$. Residual graphs for intervals $n \in [2000, 4000]$ and $n \in [3500, 4000]$ are presented in Figure 1.

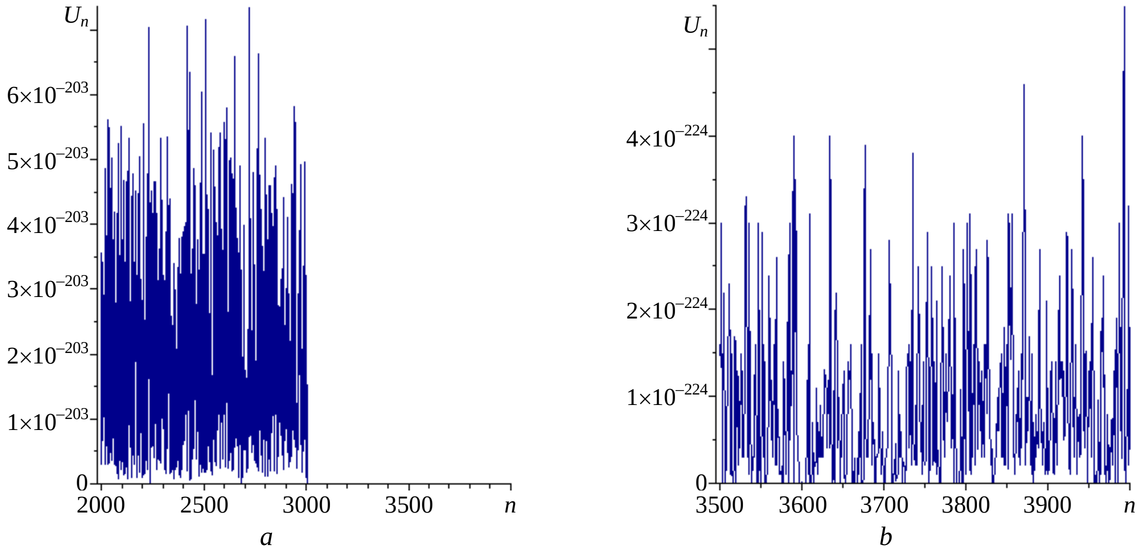

Fig. 1. Residual graphs U_n at different intervals of the variable n in the plane (n, U_n) :
$n \in [2000, 4000]$ (a); $n \in [3500, 4000]$ (b)

Now we check the periodicity condition $|x_n - x_{n+T}| + |y_n - y_{n+T}| \sim \delta$, $n > 3500$ (the computer produces zeros). We calculate the multipliers, i.e. the Jacobi matrix eigenvalues $\prod_{j=1}^{T} J(x_{5000-j}, y_{5000-j})$, where $J(x, y) = \begin{pmatrix} a|x|/x & b \\ 1 & 0 \end{pmatrix}$. We obtain $\mu_1 \approx -8.1 \cdot 10^{202}$, $\mu_2 \approx 10^{-22}$, and now are convinced that the conditions (8) are fulfilled. Thus, there is reason to believe that the orbit found really represents a cycle of length 1001. In Fig. 2 blue color shows the cycle, gray color shows the attractor.

Let we expose the coordinates of that 1001-cyclic point (rounded to the tenth sign for clarity): $x_{3500} = -0.0089836476\ldots$ $y_{3500} = -0.6850989482\ldots$

If we take another initial point, for example, $x_0 = 0$, $y_0 = 0$, we can find another cycle of length 1001. Here with the same parameter ϑ value, conditions (6) and (7) will be executed after 50 steps. To understand that the found cycle differs from the previous one, it is enough to compare the multipliers: in the second case they are equal: $\mu_1 \approx -6.5 \cdot 10^{202}$, $\mu_2 \approx -10^{-22}$.

As a result of numerous computational experiments, an interesting phenomenon was discovered that allows quickly finding the desired value for the control parameter. Namely, some parameter value is taken and a sufficiently large number of iterations are performed. Suppose that conditions (6), (7) are not satisfied. However, the eigenvalues of the matrix $\prod_{j=1}^{T} J(x_{N-j}, y_{N-j})$ are computed, where N is a sufficiently large number. Of course, these eigenvalues are not equal to the multipliers. In all the cases considered, one eigenvalue was large by its modulus the other was close to zero. The dominant eigenvalue modulus was taken as the new value of the control parameter. After the iterative procedure was run again, the sequence $\{x_n\}$ converged to a loop.

Now we'll consider an example for calculating a loop of length 1111. Take, as in the previous example, the value of the parameter $\vartheta = 10^{203}$ and the initial point $x_0 = 0$, $y_0 = 0$. We choose the accuracy of $\delta = 10^{-255}$. Now we carry out 7000 iterations. There is no convergence to the cycle. Next, we

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calculate the corresponding matrix $\prod_{j=1}^{T} J(x_{7000-j},\ y_{7000-j})$ eigenvalues, the dominant being equal to $2.83\cdot 10^{224}$, the second is close to zero. Now assuming $\vartheta = 3\cdot 10^{224}$, we run the iterative procedure again. This time conditions (6), (7) will be fulfilled after 50 steps already. The cycle multipliers are equal: $\mu_1 \approx -3.3\cdot 10^{224}$, $\mu_2 \approx -10^{-31}$, conditions (8) also hold. This means that the cycle of $T = 1111$ length is found: shown in blue at Fig. 3.

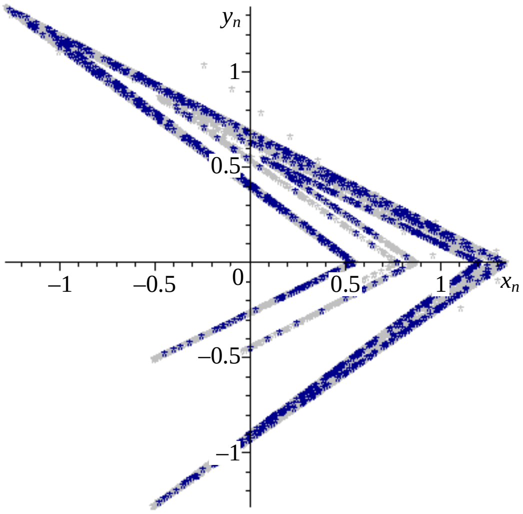

Fig. 2. 1001-cycle of the-system (9) on plane (x_n, y_n)

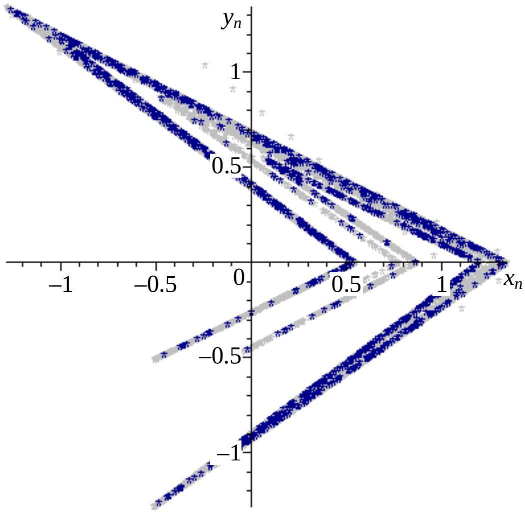

Fig. 3. 1111-cycle of the-system (9) in the plane (x_n, y_n)

A similar rule works exactly the same for Henon mapping. For other mappings, the rule had to be applied several times.

In the article [8] the examples of found mega and giga cycles for Lozi and Henon systems were given with indication to the difficulties of these cycles calculation. In particular, due to both: algorithms imperfection and insufficient computational power of computers. Unfortunately, the values of these cycles' multipliers were not given therein as well as not specified is the used accuracy of calculations.

The algorithm given in this article with the phenomenological rule of control parameter choice allows us to hope for the possibility of finding a large number of giga and even tera cycles. One of the algorithm's significant advantages refers to its insensitivity to rounding errors (due to the cycle's local asymptotic stability).

2. Henon mapping

$$x_{n+1} = 1 + a\,x_n^2 + y_n,\ \ y_{n+1} = b\,x_n,\ \ a = -1.4,\ \ b = 0.3. \qquad (10)$$

Choose $T = 1001$, $\vartheta = 5\cdot 10^{174}$, $\delta = 10^{-195}$, $x_0 = 0$, $y_0 = 0$. Now we determine that when $n > 1100$ the residual value $U_n \sim \delta$ and $|x_n - x_{n+T}| + |y_n - y_{n+T}| \sim \delta$. The residual graphs for the intervals $n \in [200, 1400]$ and $n \in [2000, 4000]$ are shown in Fig. 4.

Multipliers $\mu_1 \approx -4\cdot 10^{174}$, $\mu_2 \approx -10^{-21}$, and conditions (8) are satisfied.

1001-cyclic point: $x_{2000} = -0.6343046101\ldots$ $y_{2000} = -0.188573545\ldots$ In Fig. 5 blue color shows the cycle, gray color shows the attractor.

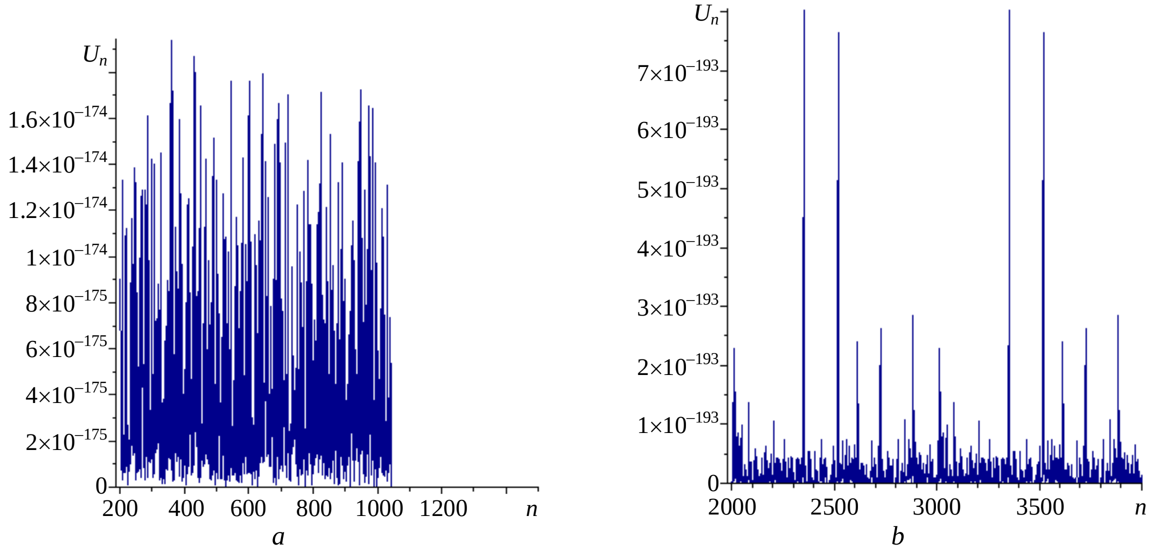

Fig. 4. Residual graphs U_n at different intervals of the variable n in the plane $(n,\ U_n)$: $n \in [200,\ 1400]$ (a); $n \in [2000,\ 4000]$ (b)

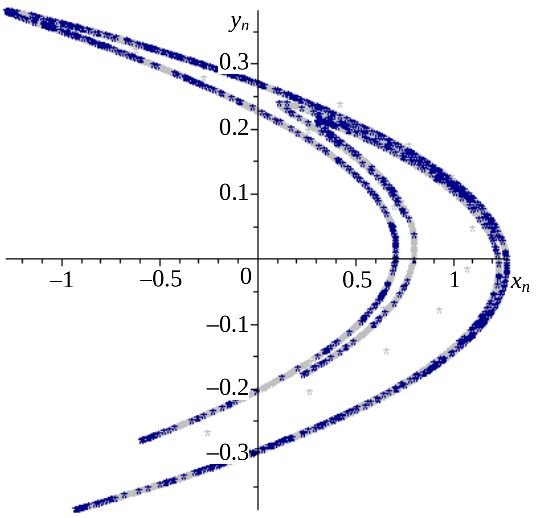

Fig. 5. 1001-cycle of the-system (10) in plane $(x_n,\ y_n)$

3. Ikeda mapping

$$x_{n+1} = 1 + u\,(x_n \cos \tau_n - y_n \sin \tau_n)\,,\quad y_{n+1} = u\,(x_n \sin \tau_n + y_n \cos \tau_n)\,, \tag{11}$$

where $u = 0.9$, $\tau_n = 0.4 - \dfrac{6}{1 + x_n^2 + y_n^2}$.

Choose $T = 1001$, $\vartheta = 3.8 \cdot 10^{225}$, $\delta = 10^{-255}$, $x_0 = 0$, $y_0 = 0$. Now we determine that when $n > 1100$ the residual value $U_n \sim \delta$ and $|x_n - x_{n+T}| + |y_n - y_{n+T}| \sim \delta$. The residual graphs for the intervals $n \in [600,\ 1500]$ and $n \in [1500,\ 2100]$ are shown in Fig. 6.

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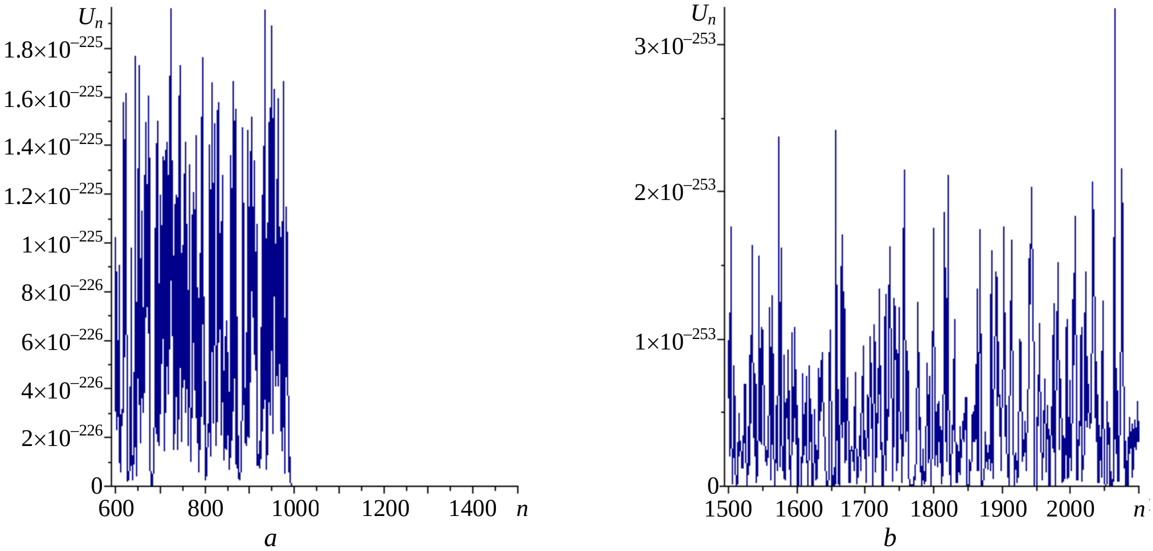

Fig. 6. Residual graphs U_n at different intervals of the variable n in the plane (n, U_n) :
$n \in [600, 1500]$ (a); $n \in [1500, 2100]$ (b)

Multipliers $\mu_1 \approx -4.8 \cdot 10^{225}$, $\mu_2 \approx 10^{-29}$, and conditions (8) are satisfied.

1001-cyclic point: $x_{2000} = 1.387200773\ldots$ $y_{2000} = 0.4458626040\ldots$ In Fig. 7 blue color shows the cycle, gray color shows the attractor.

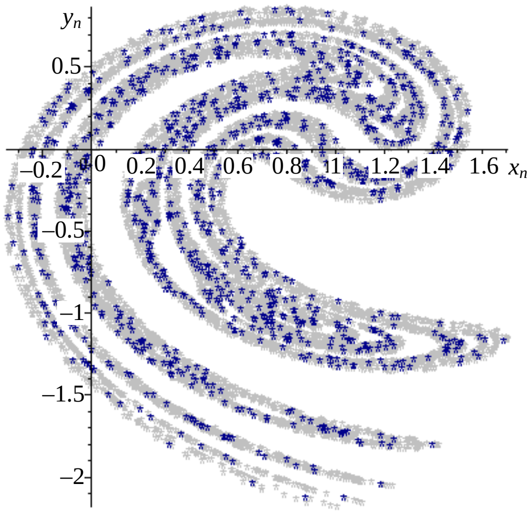

Fig. 7. 1001-cycle of the-system (11) in plane (x_n, y_n)

4. Elhadj-Sprott Mapping

$$x_{n+1} = 1 + a \sin x_n + b\, y_n ,\ y_{n+1} = x_n ,\ a = -4.0 ,\ b = 0.9 . \qquad (12)$$

Choose $T = 1001$, $\vartheta = 1.5 \cdot 10^{317}$, $\delta = 10^{-355}$, $x_0 = 0$, $y_0 = 0$. Now we determine that when $n > 500$ the residual value $U_n \sim \delta$ and $|x_n - x_{n+T}| + |y_n - y_{n+T}| \sim \delta$. The residual graphs for the intervals $n \in [50, 500]$ and $n \in [270, 290]$ are shown in Fig. 8.

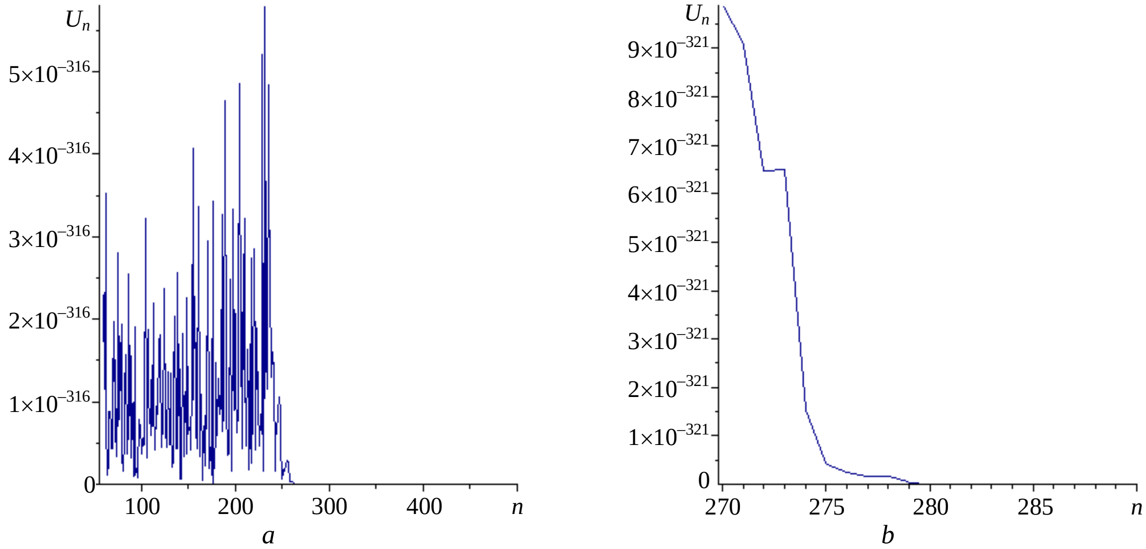

Fig. 8. Residual graphs U_n at different intervals of the variable n in the plane $(n,\ U_n)$:
$n \in [50,\ 500]$ (a); $n \in [270,\ 290]$ (b)

Multipliers $\mu_1 \approx -1 \cdot 10^{317}$, $\mu_2 \approx 0$, and conditions (8) are satisfied.

1001-cyclic point: $x_{500} = 12.6040804877\ldots$ $y_{500} = 10.2905345783\ldots$ In Fig. 9 blue color shows the cycle, gray color shows the attractor

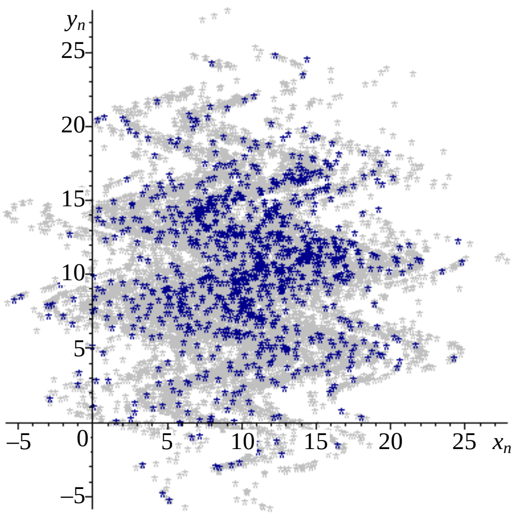

Fig. 9. 1001-cycle of the-system (12) in the plane $(x_n,\ y_n)$

5. Multihorseshoe mapping

$$x_{n+1} = x_n\, e^{a-0.8x_n-0.2y_n}\ ,\ \ y_{n+1} = y_n(0.2x_n + 0.8y_n)e^{b-0.2x_n-0.8y_n}\ ,\ a=3\ ,\ b=3\ . \qquad (13)$$

Choose $T = 1001$, $\vartheta = 1.5 \cdot 10^{187}$, $\delta = 10^{-225}$, $x_0 = 0$, $y_0 = 0$. Now we determine that when $n > 2500$ the residual value $U_n \sim \delta$ and $\left|x_n - x_{n+T}\right| + \left|y_n - y_{n+T}\right| \sim \delta$. The residual graphs for the intervals $n \in [1000,\ 3000]$ and $n \in [2500,\ 3500]$ are shown in Fig. 10.

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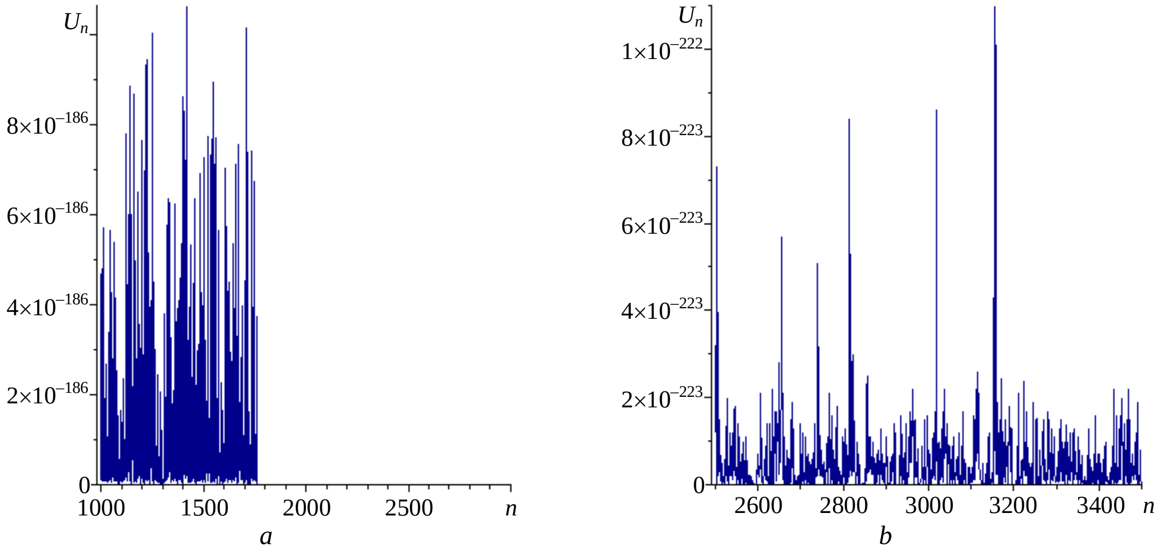

Fig. 10. Residual graphs U_n at different intervals of the variable n in the plane (n, U_n) :
$n \in [1000, 3000]$ (a); $n \in [2500, 3500]$ (b)

Multipliers $\mu_1 \approx -6.9 \cdot 10^{186}$, $\mu_2 \approx -2 \cdot 10^{-38}$, and conditions (8) are satisfied.

1001-cyclic point: $x_{2500} = 1.75599557836\ldots$ $y_{2500} = 1.70868429528\ldots$ In Fig. 11 blue color shows the cycle, gray color shows the attractor.

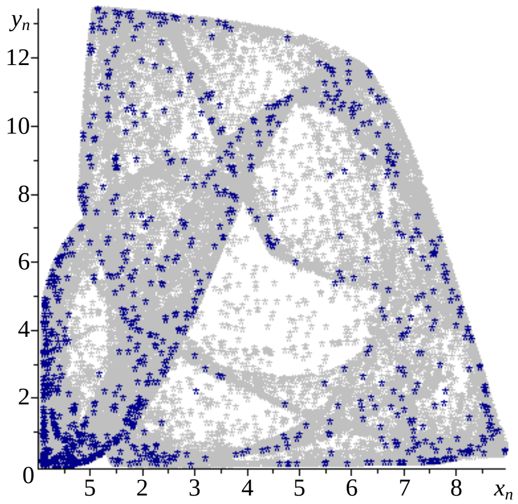

Fig. 11. 1001-cycle of the-system (13) in plane (x_n, y_n)

6. Predator-Prey mapping

$$x_{n+1} = x_n \exp(a(1 - x_n) - by_n)\,,\ x_{n+1} = x_n(1 - \exp(-cy_n))\,,\ a = 3\,,\ b = 5\,,\ c = 5\,. \qquad (14)$$

This mapping differs in its properties from the mappings discussed above. Along with dominant multiplier, containing cycles, it also has cycles whose both multipliers are large. And, judging by the numerical results, their share is significant. Therefore, to find long cycles with a dominant multiplier, the control parameter ϑ grid has been assigned and 8000 iterations were checked for each parameter value on the grid. In such a way, size cycles up to $T = 325$ have been found.

Let $T = 325$, $\vartheta = 3.8 \cdot 10^{31}$, $\delta = 10^{-45}$, $x_0 = 1$, $y_0 = 0.05$. When $n > 2600$ the residual value $U_n \sim \delta$ and $|x_n - x_{n+T}| + |y_n - y_{n+T}| \sim \delta$. The residual graphs for the intervals $n \in [600,\ 2000]$ and $n \in [2000,\ 3500]$ are shown in Fig. 12.

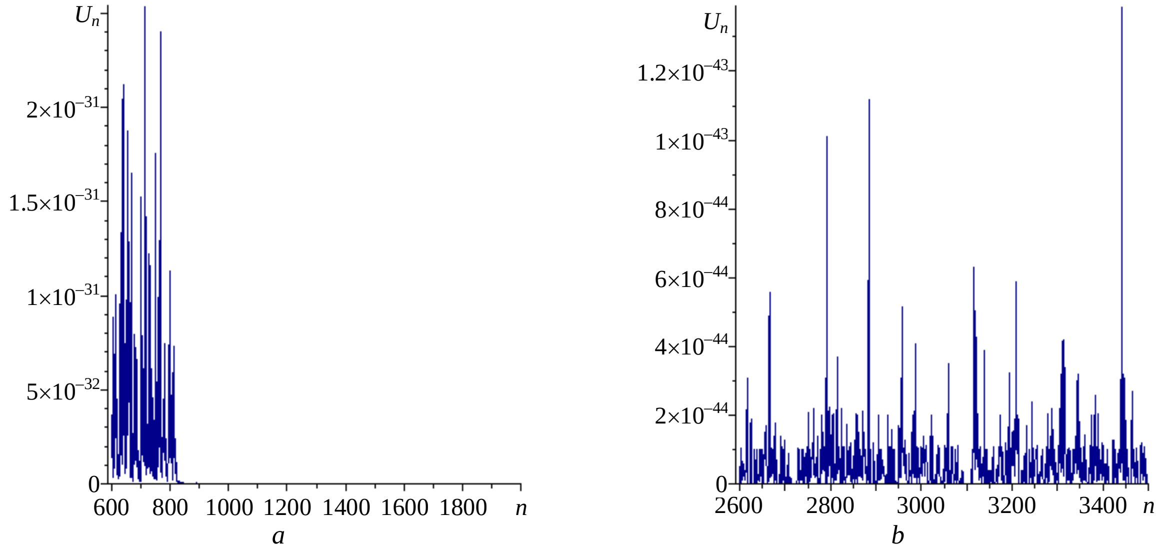

Fig. 12. Residual graphs U_n at different intervals of the variable n in the plane $(n,\ U_n)$: $n \in [600,\ 2000]$ (a); $n \in [2000,\ 3500]$ (b)

Multipliers $\mu_1 \approx -2.5 \cdot 10^{31}$, $\mu_2 \approx 0.009$, and conditions (8) are satisfied.

1001-cyclic point: $x_{2600} = 0.23346669403\ldots$ $y_{2600} =$ $0.05537511191\ldots$ In Fig. 13 blue color shows the cycle, gray color shows the attractor.

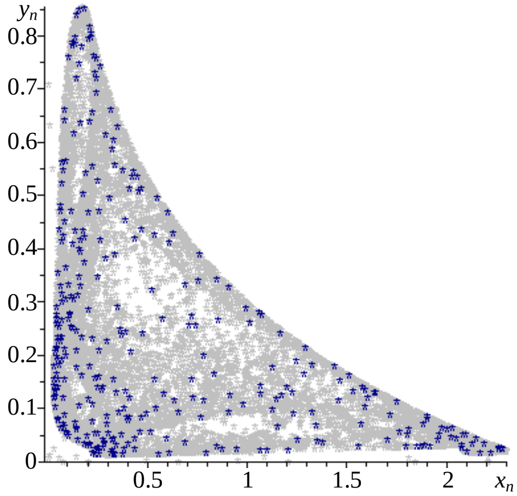

Fig. 13. 325-cycle of the-system (14) in the plane $(x_n,\ y_n)$

Conclusions

The article demonstrates the efficiency of large-length cycles searching method based on the stabilization of unstable and a priori unknown periodic orbits of these systems, using several well-known examples of nonlinear systems with discrete time. The averaged predictive control method was chosen as a stabilization method [1], which embodies the development of the predictive feedback method first proposed for discrete systems in [17].

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To improve the numerical results' reliability there were introduced the three criteria necessary for the found orbit actually was a *T*-cycle of the original dynamical system. These criteria are easily verifiable.

The proposed algorithm can be recommended for the study of the discrete dynamical systems' topological properties dependence on the change of parameters, as well as for the study of bifurcations presence and their types. Of particular interest [31] is the question about the presence of long cycles with small modulus multipliers (or even stable cycles). The averaged predictive control is particularly effective when searching for long cycles with large modulo multipliers. However, numerical simulation shows the possibility of strengthening the averaged predictive control method through synthesizing it with the method of averaged delayed control [32].

Also interesting is the problem of finding periodic orbits of continuous systems. For such systems, examples of the predictive control method application can be found in [18, 33].

Our research was mainly focused on the Lozi system, which represents one of the most studied models of systems with a strange attractor. Despite the triple control of numerical results, their reliability question, generally speaking, can never be solved definitely without ambiguity. First of all, because of computer technology imperfection, given the need to use ultra-high accuracy of data representation and very large parameters. Therefore, there is a need to test the method on supercomputers. The method correctness being confirmed, it will be interesting to find tera cycles with both large and small multipliers.

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Скринник Іван Михайлович; Skrinnik Ivan, ORCID: https://orcid.org/0000-0003-0370-6677
Дмитришин Дмитро Володимирович; Dmytryshyn Dmytro, ORCID: https://orcid.org/0000-0002-2291-2364
Стоколос Олександр Михайлович; Stokolos Alexander, ORCID: https://orcid.org/0000-0002-6632-3078
Якоб Іонут Еміл; Iacob Ionut Emil, ORCID: https://orcid.org/0000-0003-3641-0194

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