Ministry of Education and Science of Ukraine
THE ODESSA NATIONAL
POLYTECHNICAL UNIVERSITY

THE HIGHER MATHEMATICS

METHODICAL INSTRUCTIONS
for self-preparation to a practical training according to the section "Vector Algebra"

for full-time tuition students on direction
6.050503 – Engineering

Odessa: ONPU, 2015

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Укладач: І. І. Становська, к.т.н.

CONTENTS

INTRODUCTION

When studying natural sciences it is often necessary to deal with such concept as vector size or it is simple – a vector. The nobility, what is it and ability to work with vectors (or it is possible to tell in a different way: to know bases of vector algebra) is the main condition of success in studying of any discipline where vector sizes meet.

Without knowledge of vector algebra can't there be a speech about deep understanding of many sections of physics in general. Without this knowledge and it isn't necessary to speak about the correct solution of tasks.

The vector algebra is the base on which all building of classical physics is built. After all the physicist I. Newton (namely it is a basis of all studied physical disciplines) on the being is vector!

Knowledge of vector algebra is a key with which doors both in Mechanics, and in Electricity, both in Magnetism, and in any discipline where vector sizes appear open.

1 VECTORS. LINEAR OPERATIONS ON VECTORS

1.1 General provisions

Determination 1. The length of a straight line is called the direction of the segment, or vector, if you specify which of its ends is considered the beginning of the segment, and what – the end.

The vector with its origin at point *A* and ending at point *B* denoted **AB**. Often, the vector designated by a single letter (usually small Latin) **a**.

The length of the vector **AB** called his unit and designated |**AB**|. If the vector is indicated by a single letter, e.g., **b**, it is designated in the module |**b**|.

Determination 2. Two vectors **AB** and **CD** in the same direction or the same direction (in opposite directions), if they lie on parallel lines and their end points *B* and *D* are in the same half-plane (different half-planes) formed by a straight line passing through the points *A* and *C*.

Vectors **AB** and **CD** (Fig. 1) in the same direction, and the vectors **AB** and **MN** in opposite directions.

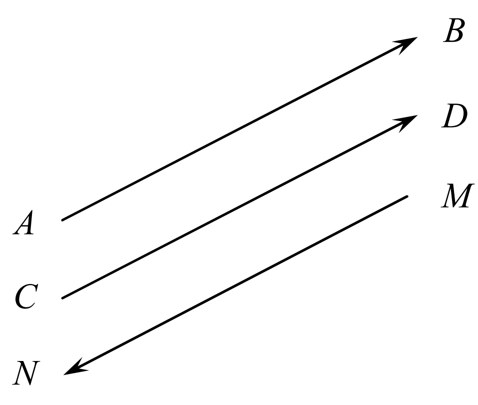

Fig. 1

If the vectors **AB** and **CD** in the same direction, it is written **AB** ↑↑**CD**, and if the opposite – **AB**↑↓**CD**.

Determination 3. Two vectors are said to be equal if they have the same length and the same direction.

Equality vectors written as **AB** = **CD**.

This definition of equality of vectors means, essentially, that the point of application of the vectors can be selected anywhere. In this sense, vectors, we will they do in the course of mathematics, called the free (in the course of mechanics, for example, along with the free study vectors and sliding vector).

Determination 4. Vector lying on the same line or parallel lines, called **collinear.**

Any two from three vectors, represented on a picture to 2 collinear each other.

If the vectors **AB** and **KL** are collinear, it is recorded **AB** || **KL**.

Determination 5. If the beginning of the vector coincides with the end of it, then this is called the zero vector and is denoted **O**. Zero vector has no direction, |**O**| = 0.

Vectors lying in one plane or in parallel planes, called **coplanar.**

1.2 The operations of vector addition and vector by a number multiplication

Determination 6. Let **a** and **b** – two vectors. If an arbitrary point O to build the vector $\mathbf{a}_1$ is **a**, then by the end of this vector to construct a vector $\mathbf{b}_1$, equal to **b** (Fig. 2), the vector extending from the beginning of the $\mathbf{a}_1$ to end $\mathbf{b}_1$, and there is **a** + **b** (rule of the "triangle").

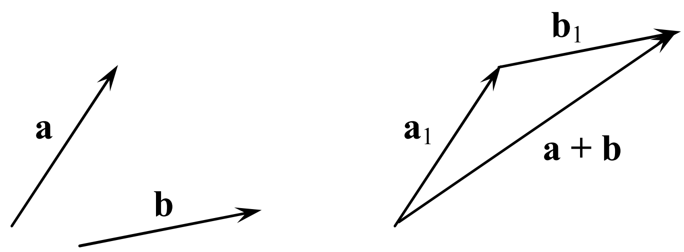

Fig. 2

In the case of non-collinear vectors **a** and **b** in order to find their sum can be used in place of the rules of the triangle parallelogram rule: should be moved to the beginning of the vectors **a** and **b** in a common point O, to build on these vectors, both the adjacent sides of the parallelogram (Fig 3); Then the vector extending from the point O to the opposite vertex of the parallelogram is **a** + **b**.

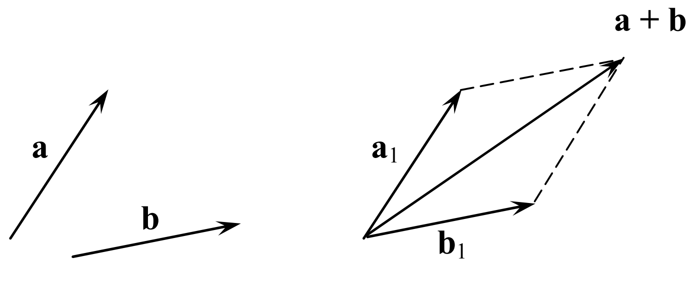

Fig. 3

It is important to understand that the constructed by the triangle rule or the rule of the parallelogram vector does not depend on the position of the point O, and the choice of the rules of addition.

Recall properties of vectors addition.

1. **a** + **b** = **b** + **a**

2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
3. $\mathbf{a} + \mathbf{o} = \mathbf{o} + \mathbf{a} = \mathbf{a}$

To construct the sum $\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n$ most convenient to use the rule–circuit breaking, a generalization of rules representing a triangle: at the end of $\mathbf{a}_1$ should be accompanied by the vector $\mathbf{a}_2$, at the end of $\mathbf{a}_2$ – vector $\mathbf{a}_3$, etc.; then the vector extending from the beginning to the end of the $\mathbf{a}_1$ and $\mathbf{a}_n$ is the sum of $\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n$ (Fig. 4)

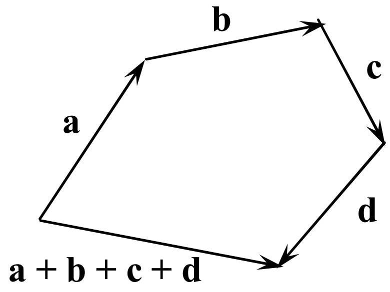

Fig. 4

Determination *7*. Given a vector **a** and the number of λ. Product $\lambda\mathbf{a}$ is the vector **b**, such that

1. $|\mathbf{b}| = |\lambda| \cdot |\mathbf{a}|$
2. $\mathbf{b} \uparrow\uparrow \mathbf{a}$, if $\lambda > 0$,
   $\mathbf{b} \uparrow\downarrow \mathbf{a}$, if $\lambda < 0$,
   $\mathbf{b} = 0$, if $\lambda = 0$.

Fig. 5 shows various cases of multiplication of vector by a number λ.

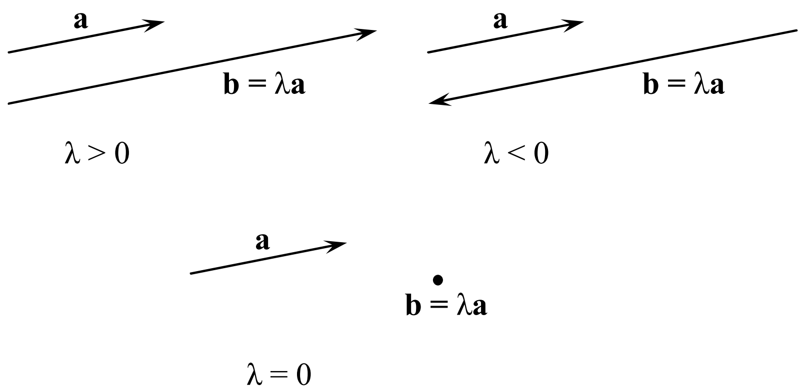

Fig. 5

Determination 8. Vector $-\mathbf{a}$ called opposite **a** if

1. $|-\mathbf{a}| = |\mathbf{a}|$
2. $-\mathbf{a} \uparrow\downarrow \mathbf{a}$

Easy to understand that the opposite vector $-\mathbf{a} = (-1) \cdot \mathbf{a}$.
Addition and multiplication by a number of vectors have the following properties:
4. $\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$
5. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

6. $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$
7. $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$
8. $0 \cdot \mathbf{a} = \mathbf{a} \cdot 0 = \mathbf{0}$

1.3 Subtraction of vectors is defined as the inverse operation to addition

Determination 9. The difference of two vectors **a** and **b** (record **a** – **b**) is called a vector **d**, which when folded with the vector **b**, gives the vector **a** (Fig. 6)

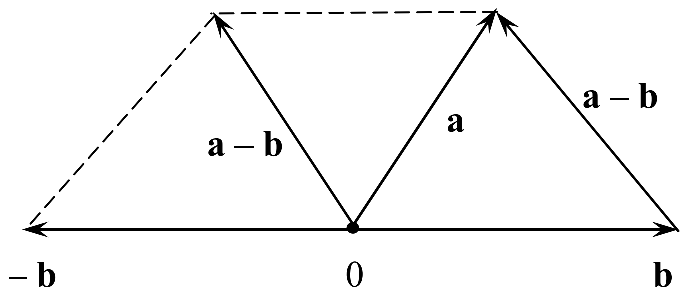

Fig. 6

We see that $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-1) \cdot \mathbf{b}$.

2 DECOMPOSITION THEOREMS

2.1 Let $\mathbf{a} \neq \mathbf{0}$, **b** – an arbitrary vector, then to $\mathbf{a} \| \mathbf{b}$ is necessary and sufficient that there exists a single number x, is $\mathbf{b} = x\mathbf{a}$

$$\left(\begin{array}{l} \mathbf{a} \neq \mathbf{o}, \forall \mathbf{b} \\ \mathbf{a} \,\|\, \mathbf{b} \Leftrightarrow \exists x \quad \mathbf{b} = x\mathbf{a} \end{array} \right)$$

The set of all vectors collinear **a**, can be seen as a set of vectors that belong to the same straight line parallel to the vector **a**. We denote the set of vectors of the letter L_1.

From the above theorem that to describe all direct vectors only one nonzero vector of this line.

2.2 Consider two non-collinear vectors **a** and **b**. Part **c** – arbitrary vector, then to a triple of vectors **a**, **b**, **c** were coplanar if and only if there exists a unique pair of numbers (x, y) such that $\mathbf{c} = x\mathbf{a} + y\mathbf{b}$

$$\left(\begin{array}{l} \mathbf{a} \,\|\, \mathbf{b}, \quad \forall \mathbf{c} \\ \mathbf{a}, \mathbf{b}, \mathbf{c} - \text{complanar} \Leftrightarrow \exists (xy) \mathbf{c} = x\mathbf{a} + y\mathbf{b} \end{array} \right)$$

The set of vectors each of which is coplanar with the vectors **a** and **b** are denoted by the letter L_2. This set can be viewed as a set of vectors lying in a plane parallel to the vectors **a** and **b**.

It follows from the theorem that all vectors to describe the plane enough to have a pair of non-collinear vectors of the plane.

2.3 The set of vectors of the entire space is denoted by the letter L_3

Suppose that **a**, **b**, **c** – triple of non-coplanar vectors space, then for any vector space **d** there is only three numbers (x, y, z) such that $\mathbf{d} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$, to describe all the vectors of three-dimensional space is enough to have some three non-coplanar vectors.

3 THE LINEAR SPACE. LINEARLY DEPENDENT AND LINEARLY INDEPENDENT SYSTEM OF VECTORS

3.1 Consider the set L of some elements, which will be called the elements of the linear space or vectors. The set L is called a linear space if the elements of this set defines two operations: addition and multiplication by a number satisfying the properties of addition and multiplication by a number of vectors (see section 1).

3.2 Let there be a system of vectors system $\mathbf{l}_1, \mathbf{l}_2,\ldots,\mathbf{l}_n$ and as many numbers $\alpha_1, \alpha_2,\ldots, \alpha_n$. Expression $\alpha_1\mathbf{l}_1 + \alpha_2\mathbf{l}_2 +\ldots+ \alpha_n\mathbf{l}_n$ will be called a linear combination $\mathbf{l}_1, \mathbf{l}_2,\ldots,\mathbf{l}_n$ with coefficients $\alpha_1, \alpha_2,\ldots, \alpha_n$.

Determination 10. System vectors $\mathbf{l}_1, \mathbf{l}_2,\ldots,\mathbf{l}_n$ called linearly independent if the linear combination is equal to **O** if and only if all of its coefficients is zero, i.e. when the relation

$$\sum_{i=1}^{n} \alpha_i \cdot \mathbf{l}_i = \mathbf{O} \Leftrightarrow \sum_{i=0}^{n} \alpha_i^2 = 0$$

Otherwise, the system of vectors is linearly dependent.

3.3 ***Determination 11***. $\mathbf{l}_1, \mathbf{l}_2,\ldots,\mathbf{l}_n$ ordered system of vectors is called a basis of the linear space if the system itself is linearly independent and every vector space can be expressed uniquely as a linear combination of vectors of this system. Number n is the dimension of the linear space. Sets L_1, L_2, L_3, reviewed in section 2, give us examples of linear spaces.

In the linear space L_2 (the plane) as a basis we can take any pair of non–collinear vectors belonging L_2. The dimension of the space is equal to two.

In the linear space L_3 as a basis we can take any three non–coplanar vectors. The dimension of the space is equal to three.

4 CARTESIAN COORDINATE SYSTEM IN THE PLANE AND IN SPACE

4.1 On the plane we choose an orthonormal basis (basis vectors have unit length and are perpendicular). Let basis vectors **i** , **j**. Both basic vectors reduced to a common beginning (at O). Axis Ox select as directed parallel to the unit vector **i**, axis Oy – parallel to the unit vector **j** (Fig. 7).

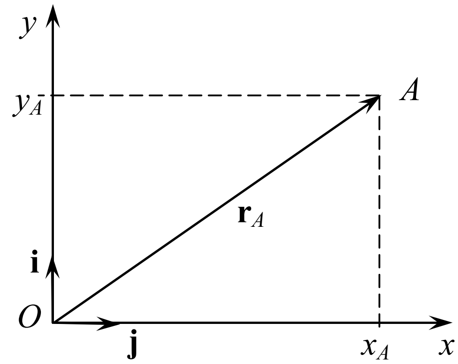

Fig. 7

Each point A of the plane put under its radius – vector $\mathbf{r}_A$ (vector connecting points O and A). By the theorem of decomposition $\mathbf{r}_A = x_A\mathbf{i} + y_A\mathbf{j}$, ie

$$A \leftrightarrow \mathbf{r}_A \leftrightarrow (x_A, y_A).$$

The pair of numbers (x_A, y_A) is called the rectangular cartesian coordinates of the point A.

4.2 It is easy to see (Fig. 7), that

$$\mathbf{r}_A = \text{pr}_{Ox}\, \mathbf{r}_A + \text{pr}_{Oy}\, \mathbf{r}_A = \text{pr}_{Ox}\, \mathbf{r}_A \cdot \mathbf{i} + \text{pr}_{Oy}\, \mathbf{r}_A \cdot \mathbf{j} \Rightarrow x = \text{pr}_{Ox}\, \mathbf{r}_A, y = \text{pr}_{Oy}\, \mathbf{r}_A,$$

ie point coordinates are projections of radius – vector of this point on the coordinate axes.

4.3 Let the plane set arbitrary vector **AB**. Known coordinates of points A (x_A, y_A) and B (x_B, y_A). It is evident

$$\mathbf{AB} = \mathbf{r}_B - \mathbf{r}_A = (x_B - x_A, y_B - y_A),$$

that is to find the coordinates of **AB** requires the coordinates of the end of the vector (point B) take the coordinates of the start vector (point A) (Fig. 8).

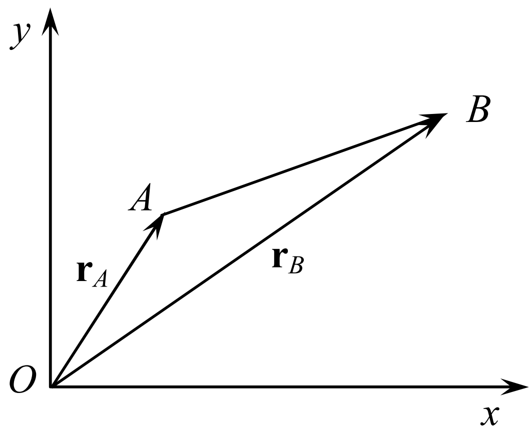

Fig. 8

<u>Comment</u>. Similarly introduced Cartesian coordinate system in space.

5 THE SCALAR PRODUCT OF VECTORS AND ITS PROPERTIES

Dot product of two vectors is a number equal to the product of the lengths of these vectors and the cosine of the angle between them.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos(\mathbf{a}, \mathbf{b})$$

Properties

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. $\mathbf{c} \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{c} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{b}$
3. $(c \cdot \mathbf{a}) \cdot \mathbf{b} = c \cdot (\mathbf{a} \cdot \mathbf{b})$
4. If one of the two vectors **a** and **b** is equal to zero, then $\mathbf{a} \cdot \mathbf{b} = 0$.

Let **a** or **b** nonzero vector, then $\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$

6 SCALAR PRODUCT IN COORDINATE FORM

Let orthonormal basis **a** = (a_x, a_y, a_z), **b** = (b_x, b_y, b_z), then

$$\mathbf{a}\cdot\mathbf{b} = a_x\cdot b_x + a_y\cdot b_y + a_z\cdot b_z$$

7 CALCULATION OF THE LENGTH OF THE VECTOR COSINE OF THE ANGLE BETWEEN THE VECTORS AND THE PROJECTION OF THE AXLE BY MEANS OF THE SCALAR PRODUCT

Let orthonormal basis **a** = (a_x, a_y, a_z), **b** = (b_x, b_y, b_z) then

$$|\mathbf{a}| = \sqrt{a^2{}_x + a^2{}_y + a_z{}^2}$$

$$\cos(\mathbf{a},\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x{}^2 + a_y{}^2 + a_z{}^2}\cdot\sqrt{b_x{}^2 + b_y{}^2 + b_z{}^2}},$$

$$\mathrm{pr}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{b_x{}^2 + b_y{}^2 + b_z{}^2}}$$

8 DETERMINANTS OF THE SECOND AND THIRD ORDER. PROPERTIES OF DETERMINANTS

8.1 Given a square array of 4 numbers $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Number $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\cdot a_{22} - a_{21}\cdot a_{12}$ is the determinant of the second order.

8.2 Given a square table of 9 numbers $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Number

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{21}a_{12}a_{33}$$

is the determinant of the third order.

The determinant of the third order can be calculated according to the following rule is easy to remember. It is necessary to write a matrix of five columns.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{pmatrix}$$

Consider the work elements on the diagonals of the arrows. We products of the elements indicated by arrows extending from the bottom up, change the sign and put all six works.

8.3 Properties of determinants.

1.The value of the determinant does not change if all lines replace columns, and each row replace the column with the same number.

2. Swapping two columns (two lines) of the determinant is equivalent to multiplying it by –1.

Corollary. If determinant has two identical rows (columns), it is equal to zero.

3. Multiplying all the elements of one column (row) of the determinant on any number *k* is equivalent to multiplying the determinant on the number *k*.

Consequences:

If all elements of a column (line) of the determinant zero, the determinant is zero.

If the corresponding elements of two rows (columns) of the determinant are proportional, then the determinant is zero.

4. If each element of a row (or column) is the sum of two terms, the determinant can be presented as a sum of two determinants, one of which is on this line (column) is first of said terms, and the other – second; elements at other places, all three determinants of the same.
For example,

$$\begin{vmatrix} a'_{11}+a''_{11} & a_{12} & a_{13} \\ a'_{21}+a''_{21} & a_{22} & a_{23} \\ a'_{31}+a''_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a'_{11} & a_{12} & a_{13} \\ a'_{21} & a_{22} & a_{23} \\ a'_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a''_{11} & a_{12} & a_{13} \\ a''_{21} & a_{22} & a_{23} \\ a''_{31} & a_{32} & a_{33} \end{vmatrix}$$

5. If the elements of a row (column) to add the corresponding elements of another row (or column) multiplied by any other common factor, the value of the determinant does not change.

6. Minor (M_{ij}) element (a_{ij}) is called the determinant of the determinant obtained from this by deleting the line with the number *i* and number *j* of the column, which is located at the intersection of this element.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

The cofactor (A_{ij}) element (a_{ij}) is equal to the determinant of the minor elements, taken with a certain mark

$$A_{ij} = (-1)^{i+j} M_{ij}$$

The determinant is the sum of products of the elements of any row (column) by their cofactors addition, for example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}\cdot A_{21} + a_{22}\cdot A_{22} + a_{23}\cdot A_{23}$$

9 VECTOR PRODUCT AND ITS PROPERTIES

9.1 Suppose we are given two vectors **a** and **b**. The vector product **a**×**b** vector **a** to vector **b** called third vector **c**, defined by the following conditions:

1.|**c**| = |**a**×**b**| = |**a**|·|**b**|·sin(a,b) = S▱
2. **c** ⊥ **a**, **c** ⊥ **b**

and triple vectors **a**, **b**, **c** taken in the written procedure, a triple orientation, ie, from the end of the third vector **c** shortest, turn on the first vector **a** to the second **b** seen counterclockwise (it is assumed that all three vectors are given to the common top) (Fig. 9).

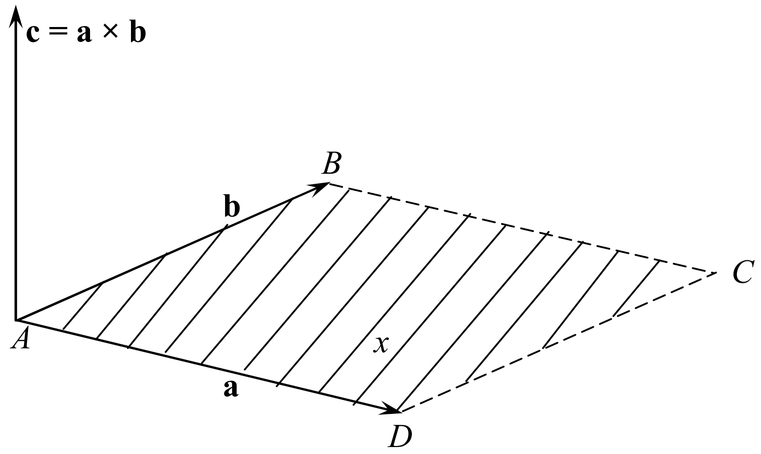

Fig. 9

In the definition of the vector product S▱ area of the parallelogram *ABCD* constructed on the vectors **a** and **b** (Fig. 12)

9.2 Properties:
1. (condition of collinearity) **a**||**b** ⇔ **a** × **b** = 0
2. **a** × **b** = – **b** × **a**
3. (λ**a**) × **b** = λ(**a** × **b**)
4. **c** × (**a** + **b**) = **c** × **a** + **c** × **b**
5. In applications for the area of a triangle formed by the vectors **a** and **b**, using the formula

$$S_{\Delta} = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

10 VECTOR PRODUCT IN COORDINATE FORM

If the vectors **a** and **b** are given in the right coordinate system (**i**, **j**, **k** basis vectors have the right orientation) its coordinates: **a** = (a_x, a_y, a_z), **b** = (b_x, b_y, b_z), the vector product **a** × **b** is determined by the formula

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = A_{11}\mathbf{i} + A_{12}\mathbf{j} + A_{13}\boldsymbol{k}$$

11 MIXED PRODUCT OF THREE VECTORS AND ITS PROPERTIES

11.1 Let three vectors **a**, **b**, **c**. Mixed product of three vectors **a**, **b**, **c** (recorded **abc**) call number which is equal to the vector product **a** × **b**, multiplied by the vector **c** , ie.

$$\mathbf{abc} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

11.2 Properties:

1. (condition of coplanarity) **a**, **b**, **c** – coplanar $\Leftrightarrow$ $\mathbf{abc} = 0$,
2. Let **a**, **b**, **c** – triple of non-coplanar vectors, then (Fig. 10)

$$\mathbf{abc} = \begin{cases} V_{\mathrm{B}}, \text{ if } \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ you have the right orientation} \\ -V_B, \text{ if } \mathbf{a}, \mathbf{b}, \mathbf{c} \text{ you have a left orientation} \end{cases}$$

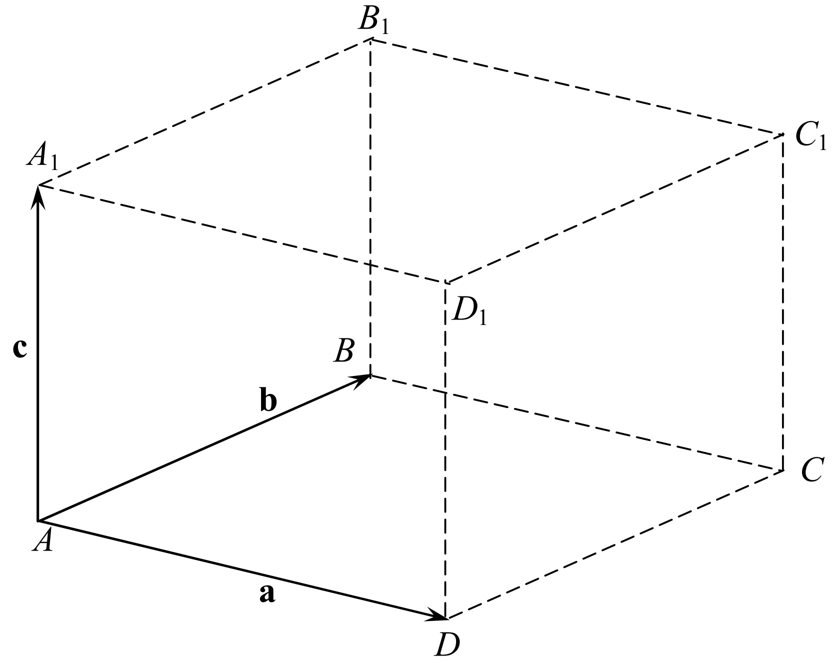

Fig. 10

In formula V_B – the volume of the box, $ABCDA_1B_1C_1D_1$ constructed on the vectors **a**, **b** and **c**.

12 THE MIXED PRODUCT IN COORDINATE FORM

If vectors **a**, **b**, **c** set in the right frame of reference for its coordinates: $\mathbf{a} = (a_x, a_y, a_z)$, $\mathbf{b} = (b_x, b_y, b_z)$, $\mathbf{c} = (c_x, c_y, c_z)$ the mixed product **abc** is given by

$$\mathbf{abc} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

13 EXAMPLES OF PROBLEM SOLVING BY VECTOR ALGEBRA

13.1 Calculate the determinant

$$\begin{vmatrix} 2 & 0 \\ -5 & 6 \end{vmatrix}$$

$$\begin{vmatrix} 2 & 0 \\ -5 & 6 \end{vmatrix} = 2 \cdot 6 - 0 \cdot (-5) = 12 - 0 = 12$$

13.2 Solve the inequality

$$\begin{vmatrix} 1 & x+5 \\ 2 & x \end{vmatrix} < 0$$

$$\begin{vmatrix} 1 & x+5 \\ 2 & x \end{vmatrix} = x - 2(x+5) = -x - 10,$$

$$-x - 10 < 0$$

$$x > -10$$

$$x \in (-10, \infty)$$

13.3 Solve the equation

$$\begin{vmatrix} 4\sin x & 1 \\ 1 & \cos x \end{vmatrix} = 0$$

$$\begin{vmatrix} 4\sin x & 1 \\ 1 & \cos x \end{vmatrix} = 4\sin x \cdot \cos x - 1 = 2\sin(2x) - 1,$$

$$\sin 2x = \frac{1}{2},$$

$$2x = (-1)^k \arcsin n \frac{1}{2} + k\pi, \quad k \in Z$$

$$2x = k\pi + (-1)^k \frac{\pi}{6}, \quad k \in Z$$

$$x = \frac{k\pi}{2} + (-1)^k \frac{\pi}{12}, \quad k = 0, = 1, \ldots$$

13.4 Calculate the determinant $\begin{vmatrix} 2 & 4 & -1 \\ 7 & 3 & 2 \\ 3 & 1 & -2 \end{vmatrix}$

$$\begin{vmatrix} 2 & 4 & -1 \\ 7 & 3 & 2 \\ 3 & 1 & -2 \end{vmatrix} = 2 \cdot 3 \cdot (-2) + 7 \cdot 1 \cdot (-1) + 4 \cdot 2 \cdot 3 - 3 \cdot 3 \cdot (-1) - 7 \cdot 4 \cdot (-2) - 2 \cdot 1 \cdot 2 =$$

$$= -12 - 7 + 24 + 9 + 56 - 4 = 66$$

13.5 Calculate the determinant $\begin{vmatrix} 2 & 0 & 5 \\ 1 & 3 & 16 \\ 0 & -1 & 10 \end{vmatrix}$,

laying it on the elements of the first column

$$\begin{vmatrix} 2 & 0 & 5 \\ 1 & 3 & 16 \\ 0 & -1 & 10 \end{vmatrix} = 2(-1)^{1+1}\begin{vmatrix} 3 & 16 \\ -1 & 10 \end{vmatrix} + 1\cdot(-1)^{1+2}\begin{vmatrix} 0 & 5 \\ -1 & 10 \end{vmatrix} + 0\cdot(-1)^{1+3}\begin{vmatrix} 0 & 5 \\ 3 & 16 \end{vmatrix} =$$
$$= 2\cdot 1(3\cdot 10 - 16\cdot(-1)) + 1\cdot(-1)\cdot(0\cdot 10 - 5\cdot(-1)) + 0 = 2\cdot 46 - 1\cdot 5 = 92 - 5 = 87$$

Let's calculate the determinant, expanding it on the elements of, for example, the third row.

$$\begin{vmatrix} 2 & 0 & 5 \\ 1 & 3 & 16 \\ 0 & -1 & 10 \end{vmatrix} = 0(-1)^{3+1}\begin{vmatrix} 0 & 5 \\ 3 & 16 \end{vmatrix} + (-1)\cdot(-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 1 & 16 \end{vmatrix} + 10\cdot(-1)^{3+3}\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} =$$
$$= 0\cdot(-1)\cdot(-1)\cdot(32-5) + 10\cdot 1(6-0) = 27 + 60 = 87$$

We see that the result is the same

13.6 Calculate the determinant $\begin{vmatrix} 5 & 3 & 2 \\ -1 & 2 & 4 \\ 7 & 3 & 6 \end{vmatrix}$ after converting it.

Let us add to the elements of the first row of the corresponding elements of the second row, multiplied by 5, and the elements of the third row – the relevant elements of the second row multiplied by 7 (under these transformations are not the determinant value changes)

$$\begin{vmatrix} 5 & 3 & 2 \\ -1 & 2 & 4 \\ 7 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 0 & 13 & 22 \\ -1 & 2 & 4 \\ 0 & 17 & 34 \end{vmatrix}$$

Section now receiving the last qualifier on the elements of the first column, we obtain

$$\begin{vmatrix} 0 & 13 & 22 \\ -1 & 2 & 4 \\ 0 & 17 & 34 \end{vmatrix} = 0 \qquad \begin{vmatrix} 2 & 4 \\ 17 & 34 \end{vmatrix} - (-1)\begin{vmatrix} 13 & 22 \\ 17 & 34 \end{vmatrix} + 0\begin{vmatrix} 13 & 22 \\ 2 & 4 \end{vmatrix} = 13\cdot 34 - 17\cdot 22 = 68$$

13.7 Calculate the length of the vector $\mathbf{a} = (2,-1,1)$ and its direction cosines

$$|\mathbf{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = 6;$$
$$\tilde{n}os\alpha = \frac{2}{6}; \qquad \cos\beta = \frac{-1}{6}; \qquad \cos\gamma = \frac{1}{6}$$

13.8 Given to $|\mathbf{a}| = 11$, $|\mathbf{b}| = 23$ and $|\mathbf{a}-\mathbf{b}| = 30$. Determine the $|\mathbf{a} + \mathbf{b}|$.

Let's vectors **a** and **b**, bringing them to the overall top. Then, vectors $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ are constructed diagonals of the parallelogram (Fig. 11). Our problem is reduced to the problem of elementary geometry: the two sides and one diagonal of the parallelogram to find the second diagonal.

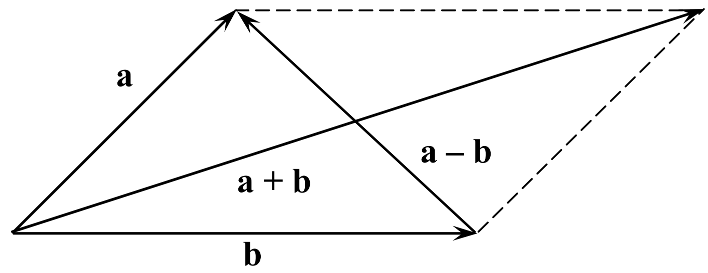

Fig. 11

Knowing that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals, we get

$$|\mathbf{a} + \mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2 = 2\cdot 11^2 + 2\cdot 23^2 - 30^2 = 400$$

that is $|\mathbf{a} + \mathbf{b}| = 20$

13.9 Given two vector $\mathbf{a} = (3; -2; 6)$ and $\mathbf{b} = (-2; 1; 0)$

1) $\mathbf{a}+\mathbf{b}$; 2) $\mathbf{a}-\mathbf{b}$; 3) $2\mathbf{a}$; 4) $2\mathbf{a}+3\mathbf{b}$

1) $\mathbf{a} + \mathbf{b} = (3 + (-2); -2 + 1; 6 + 0)$.
2) $\mathbf{a} - \mathbf{b} = (3 - (-2); -2 - 1; 6 - 0)$, $\mathbf{a} - \mathbf{b} = (5; -3; 6)$.
3) $2\mathbf{a} = (3\cdot 2; -2\cdot 2; 6\cdot 2)$, $2\mathbf{a} = (6; -4; 12)$.
4) $3\mathbf{b} = (-6; 3; 0)$, $2\mathbf{a} + 3\mathbf{b} = (6 + (-6) ; -4 + 3; 12 + 0)$, $2\mathbf{a}+3\mathbf{b} = (0; -1; 12)$.

13.10 Find unit vector $\mathbf{a} = (6; -2; -3)$.

As the unit vector **a** vector is a new vector $\mathbf{a}^0$ directed in the same way as vector **a**, with a length equal to unity, then

$$\mathbf{a}^0 = \frac{1}{|\mathbf{a}|}\cdot\mathbf{a} \quad \text{or} \quad \mathbf{a}^0 = \left(\frac{a_x}{|\mathbf{a}|}, \ \frac{a_y}{|\mathbf{a}|}, \ \frac{a_z}{|\mathbf{a}|}\right)$$

Calculating $|\mathbf{a}| = \sqrt{36 + 4 + 9} = 7$

We get the solution

$$\mathbf{a}^0 = \left(\frac{6}{7}; \frac{-2}{7}; \frac{-3}{7}\right)$$

13.11 Given two vector $\mathbf{a} = (4; -2; -4)$, $\mathbf{b} = (6; -3;\ 2)$

Calculate:

1) $\mathbf{a}\cdot\mathbf{b}$; 2) $(2\mathbf{a} - 3\mathbf{b})\cdot(\mathbf{a} + 2\mathbf{b})$

1) $\mathbf{a}\cdot\mathbf{b} = 4\cdot 6 + (-2)\cdot(-3) + (-4)\cdot 2 = 24 + 6 - 8 = 22$
2) $2\mathbf{a} = (8; -4; -8)$
$3\mathbf{b} = (18; -9; 6)$
$2\mathbf{b} = (12; -6; 4)$

$2\mathbf{a} - 3\mathbf{b} = (-10; 5; -14)$
$\mathbf{a} + 2\mathbf{b}=(16; -8; 0)$

Then

$$(2a - 3b) \cdot (a + 2b) = (-10) \cdot 16+5 \cdot (-8) + (-14)\cdot 0= -160 - 40 = -200.$$

13.12 Calculate what kind of work force produces $\mathbf{F} = (3; -5; 2)$, when its point of application is moved from the beginning to the end of the vector
$\mathbf{S} = (2; -5; -7)$

If the vector **F** is a force whose point of application moves from the start to the end of the vector **S**, the work **A** of this force is determined by the equation $\mathbf{A} = \mathbf{F} \cdot \mathbf{S}$.

So:

$$\mathbf{A} = 3\cdot 2 + (-5) \cdot (-5) + 2\cdot (-7) = 6 + 25 - 14 = 17$$

13.13 Determine at what value α vectors $\mathbf{a} = \alpha\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \alpha\mathbf{k}$ are mutually perpendicular.

Since $\mathbf{a}\cdot\mathbf{b} = 0$ if and only if $\mathbf{a} \perp \mathbf{b}$ then

$$\alpha\cdot 1 + (-3) \cdot 2 + 2\cdot (-\alpha) = 0$$
$$-6 -\alpha = 0$$
$$\alpha = -6$$

13.14 Given point $A(-2; 3; -4)$, $B(3; 2; 5)$, $C(1; -1; 2)$ and $D(3; 2; -4)$. Calculate $\mathrm{pr}_{\mathbf{CD}}\mathbf{AB}$.

$$\mathrm{pr}_{\mathbf{CD}}\mathbf{AB} = \frac{\mathbf{AB}\cdot\mathbf{CD}}{|\mathbf{CD}|}$$

We find $\mathbf{AB} = (5; -1; 9)$, $\mathbf{CD} = (2; 3; -6)$.

Then $\mathrm{pr}_{\mathbf{CD}}\mathbf{AB} = \dfrac{5\cdot 2+(-1)\cdot 3+9\cdot(-6)}{\sqrt{2^2+3^2+(-6)^2}} = -\dfrac{47}{7}$

13.15 Backgrounds **a** and **b** form an angle $\varphi = \frac{2}{3}\pi$. Knowing that $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$, calculate $|(2\mathbf{a} + \mathbf{b})\times(\mathbf{a} + 2\mathbf{b})|$.

Using the properties of the vector product, we obtain

$$(2\mathbf{a} + \mathbf{b})\times(\mathbf{a} + 2\mathbf{b})=2\mathbf{a}\times\mathbf{a} + 2\mathbf{a}\times 2\mathbf{b} + \mathbf{b}\times\mathbf{a} + \mathbf{b}\times 2\mathbf{b}=2(\mathbf{a}\times\mathbf{a}) + 4(\mathbf{a}\times\mathbf{b}) +$$
$$\mathbf{b}\times\mathbf{a} + 2(\mathbf{b}\times\mathbf{b}) = 2\cdot 0 + 4(\mathbf{a}\times\mathbf{b}) + \mathbf{b}\times\mathbf{a} + 2\cdot 0=4(\mathbf{a}\times\mathbf{b}) - (\mathbf{a}\times\mathbf{b}) = 3(\mathbf{a}\times\mathbf{b})$$

It remains to find the length of the resulting vector

$$|(2\mathbf{a} + \mathbf{b})\times(\mathbf{a} + 2\mathbf{b})| = |3(\mathbf{a}\times\mathbf{b})| = 3|(\mathbf{a}\times\mathbf{b})| = 3\cdot|\mathbf{a}|\cdot|\mathbf{b}|\cdot\sin(\mathbf{a};\mathbf{b}) = 3\cdot 1\cdot 2\cdot \sin\frac{2}{3}\pi = 6\cdot\frac{\sqrt{3}}{2} = 3\sqrt{3}$$

13.16 Force $\mathbf{P} = (2; -4; 5)$ is applied to the point $M_0(4; -2; 3)$. Determine the moment of this force about point $A(3; 2; -1)$.

If the vector **P** is a force applied to any point M, **a** is a vector from a point O to point M, then vektor $\mathbf{a}\times\mathbf{P}$ a torque **P** relative to point O.

we have **P** = (2; –4; 5) and **a** = $\mathbf{AM_0}$ = (1; –4; 4)

$$AM_0\times p = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 4 \\ 2 & -4 & 5 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -4 & 4 \\ -4 & 5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 4 \\ 2 & \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -4 \\ 2 & -4 \end{vmatrix} = -4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$$

13.17 Vectors **AB** and **CD** determine the coordinates of its ends: *A*(2; 4; 5), *B*(–1; –3; –2), *C*(4; 1; 7), *D*(–2; 3; 10). Find the area of the parallelogram constructed on the vectors **AB** and **CD**.

We find first of all, the coordinates of the vectors **AB** and **CD**:

$$AB = (-1 -2; -3 - 4; -2 - 5) = (-3; -7; -7),$$
$$CD = (-2 - 4; 3 - 1; 10 - 7) = (-6; 2; 3).$$

Next, we find the vector product **AB** and **CD** according to the formula:

$$\mathbf{AB}\times\mathbf{CD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -7 & -7 \\ -6 & 2 & 3 \end{vmatrix} = -7\mathbf{i} + 51\mathbf{j} - 48\mathbf{k}$$

It remains to find the length of the vector $\mathbf{AB}\times\mathbf{CD}$ length of the vector by the formula

$$|\,\mathbf{AB}\times\mathbf{CD}\,| = \sqrt{(-7)^2 + 51^2 + (-48)^2} = \sqrt{4954}$$

13.18 Calculate the sine of the angle formed by the vectors **a** = (2; –2; 1) and **b** = (2; 3; 6) We find the vector product **a** and **b**:

$$a\times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 2 & 3 & 6 \end{vmatrix} = -15\mathbf{i} - 10\mathbf{j} + 10\mathbf{k}.$$

Then $|\mathbf{a}\times\mathbf{b}| = \sqrt{(-5)^2 + (-10)^2 + 10^2} = \sqrt{425}$ and since

$$|\mathbf{a}| = \sqrt{2^2 + (-2)^2 + 1} = 3, \quad |\mathbf{b}| = \sqrt{2^2 + 3^2 + 6^2} = 7.$$

Then

$$\sin(\mathbf{a},\mathbf{b}) = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\cdot|\mathbf{b}|} = \frac{\sqrt{425}}{21} = \frac{5\sqrt{17}}{21}.$$

13.19 Determine whether the vectors are coplanar **a**, **b**, **c**.
a = (2; 3; –1), **b** = (1; –1; 3), **c** = (1; 9; –11).

Three vector coplanar if and only if their mixed product is zero. We find mixed product of these vectors:

$$\mathbf{abc} = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & 3 \\ 1 & 9 & -11 \end{vmatrix} = 22 - 9 + 9 - 1 - 54 + 33 = 0.$$

This means that these vectors are coplanar.

13.20 Given the coordinates of the vertices of the pyramid $A_1(5;\ 1;\ -4)$, $A_2(1;\ 2;\ -1)$, $A_3(3;\ 3;\ -4)$ and $A_4(2;\ 2;\ 2)$. Determine its volume
Consider three vectors: $\mathbf{A_1A_2}$, $\mathbf{A_1A_3}$, $\mathbf{A_1A_4}$. The volume of the pyramid, built on these vectors is equal to one-sixth of the module mixed product of these vectors. Calculating $\mathbf{A_1A_2} = (-4;\ 1;\ 3)$, $\mathbf{A_1A_3} = (-2;\ 2;\ 0)$, $\mathbf{A_1A_4} = (-3;\ 1;\ 6)$, obtain

$$\mathbf{V} = \left\| \begin{matrix} -4 & 1 & 3 \\ -2 & 2 & 0 \\ -3 & 1 & 6 \end{matrix} \right\| = \frac{1}{6} \cdot |-24| = 4$$

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