Univalent polynomials and Koebe's one-quarter theorem

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UNIVALENT POLYNOMIALS AND KOEBE'S ONE-QUARTER THEOREM

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Abstract. The famous Koebe $\frac{1}{4}$ theorem deals with univalent (i.e., injective) analytic functions f on the unit disk $\mathbb{D}$. It states that if f is normalized so that $f(0) = 0$ and $f'(0) = 1$, then the image $f(\mathbb{D})$ contains the disk of radius $\frac{1}{4}$ about the origin, the value $\frac{1}{4}$ being best possible. Now suppose f is only allowed to range over the univalent polynomials of some fixed degree. What is the optimal radius in the Koebe-type theorem that arises? And for which polynomials is it attained? A plausible conjecture is stated, and the case of small degrees is settled.

1. Introduction

Suppose you can solve a certain problem that involves general analytic functions, perhaps lying in some (fairly large) class. Then you restrict your attention to the set of polynomials of a fixed degree that are in the same class. Can you also solve the restricted (polynomial) version of the problem that arises? Well, not necessarily. While many a problem is sure to simplify or trivialize completely, there are others that become dramatically harder. In what follows, we deal with a situation of the latter kind.

Our starting point is the classical Koebe one-quarter theorem, a cornerstone of geometric function theory. Recall, first of all, that an analytic function on a domain $\Omega \subset \mathbb{C}$ is said to be *univalent* if it is one-to-one, i.e., takes distinct values at distinct points of Ω. Now let S denote the set of univalent functions f on the disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ that have a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

(so that $f(0) = 0$ and $f'(0) = 1$). The one-quarter theorem – which was actually conjectured by Koebe in 1907 and proved somewhat later by Bieberbach – reads as follows.

Theorem 1.1 (Koebe's $\frac{1}{4}$ theorem). *For every $f \in S$, the range $f(\mathbb{D})$ contains the disk $\{w : |w| < \frac{1}{4}\}$.*

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See, e.g., [7, Chapter 2] or [12, Chapter 14] for a proof. Furthermore, since the so-called *Koebe function*

$$\mathcal{K}(z) := \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots, \qquad z \in \mathbb{D}, \tag{1.2}$$

belongs to S and maps $\mathbb{D}$ onto the slit plane $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$, we see that the radius $\frac{1}{4}$ in Koebe's theorem is optimal; indeed, no larger number would do.

Motivated by this result, we introduce the following notation. Given a set $X \subset S$, we write $\rho(X)$ for the supremum of those $r > 0$ for which the common range $\bigcap_{f \in X} f(\mathbb{D})$ contains the disk $\{w : |w| < r\}$. The number $\rho(X)$ will be referred to as the *Koebe radius* for X. Clearly, we always have $\rho(X) \ge \frac{1}{4}$.

An important subclass of S, to be denoted by $S_{\mathbb{R}}$, is the set of univalent functions f of the form (1.1) whose coefficients a_n are all real. Because the Koebe function $\mathcal{K}$ is in $S_{\mathbb{R}}$, we see that

$$\rho(S_{\mathbb{R}}) = \rho(S) = \frac{1}{4}.$$

We further remark that the critical value $\frac{1}{4}$ coincides with

$$|\mathcal{K}(-1)| = \min\{|\mathcal{K}(\zeta)| : \zeta \in \mathbb{T}\}, \tag{1.3}$$

where $\mathbb{T} := \partial\mathbb{D}$ is the unit circle, and moreover, -1 is the only minimum modulus point for $\mathcal{K}$ on $\mathbb{T}$.

Among the many results that highlight the extremal role of the Koebe function $\mathcal{K}$ in S and/or $S_{\mathbb{R}}$, the most famous is undoubtedly the (former) *Bieberbach conjecture*, now de Branges' theorem, which establishes sharp bounds for the coefficients a_n in (1.1). Namely, it states that every $f \in S$ (and hence every $f \in S_{\mathbb{R}}$) satisfies $|a_n| \le n$ for $n = 2, 3, \dots$, the inequalities being all strict unless f is the Koebe function $\mathcal{K}$ or one of its rotations. (In the case of $S_{\mathbb{R}}$, the only nontrivial rotation to be considered is $z \mapsto -\mathcal{K}(-z)$.)

We mention in passing that the $S_{\mathbb{R}}$ version of the Bieberbach conjecture was relatively easy to settle; one of the proofs (as outlined in [7, p. 269]) makes use of Suffridge's work on univalent polynomials, a topic to be touched upon below. By contrast, the full version of the conjecture had remained open for almost 70 years, defying numerous attacks, until de Branges finally cracked it by using a highly sophisticated array of techniques (see [3, 8]).

It is noteworthy that the two extremal problems – those underlying the Koebe $\frac{1}{4}$ theorem and the Bieberbach conjecture – are tightly linked together. In fact, the basic inequality $|a_2| \le 2$, which was discovered by Bieberbach in 1916, both led him to a proof of Theorem 1.1 and provided the basis for his coefficient conjecture.

Now, we are interested in polynomial versions of Theorem 1.1. The extremal problem that arises, to be described in a moment, will be referred to as the *polynomial Koebe problem.* Given a positive integer N, let $\mathcal{U}_N$ (resp., $\mathcal{U}_{N,\mathbb{R}}$) denote the set of univalent polynomials p of the form

$$p(z) = z + \sum_{n=2}^{N} a_n z^n \tag{1.4}$$

with complex (resp., real) coefficients; here and below, univalence is only assumed in $\mathbb{D}$. Thus, $\mathcal{U}_N \subset S$ and $\mathcal{U}_{N,\mathbb{R}} \subset S_{\mathbb{R}}$. We then ask: *What are the values of the Koebe radii* $\rho(\mathcal{U}_N)$ *and* $\rho(\mathcal{U}_{N,\mathbb{R}})$*?* Also, *what are the extremal "Koebe-type" polynomials that minimize the quantity*

$$\text{dist}\,(p(\mathbb{T}), \{0\}) := \min\{|p(\zeta)| : \zeta \in \mathbb{T}\} \tag{1.5}$$

among all p *in* $\mathcal{U}_N$ *and/or* $\mathcal{U}_{N,\mathbb{R}}$*?*

Clearly, the optimal lower bound for (1.5), as p ranges over $\mathcal{U}_N$ or $\mathcal{U}_{N,\mathbb{R}}$, coincides with the Koebe radius of the corresponding class. Once again, the case of real coefficients seems to be more tractable, so we restrict most of our attention to $\mathcal{U}_{N,\mathbb{R}}$.

Needless to say, the problem is trivial for $N = 1$. Indeed, the only element of $\mathcal{U}_1$ (as well as of $\mathcal{U}_{1,\mathbb{R}}$) is the identity function z, whence

$$\rho\,(\mathcal{U}_1) = \rho\,(\mathcal{U}_{1,\mathbb{R}}) = 1.$$

The case $N = 2$ is not much harder. This time, the univalent polynomials of the form $z + a_2 z^2$ are precisely those with $|a_2| \le \frac{1}{2}$; the extremal ones have $|a_2| = \frac{1}{2}$, and a simple calculation shows that

$$\rho\,(\mathcal{U}_2) = \rho\,(\mathcal{U}_{2,\mathbb{R}}) = \frac{1}{2}.$$

Typically enough, passing to higher degrees makes life increasingly painful, and the case of $N = 3$ already seems to deserve a serious analysis. To begin with, it is far from trivial to determine the values of a_2 and a_3 for which the polynomial $z + a_2 z^2 + a_3 z^3$ is univalent in $\mathbb{D}$. This has been done, however, and we recall the result (or rather its $\mathcal{U}_{3,\mathbb{R}}$ version) in Section 4 below. Then we proceed to solve our polynomial Koebe problem for $\mathcal{U}_{3,\mathbb{R}}$.

Meanwhile, we pause to speculate on the case of general N, trying to guess what the extremal polynomials should be. One natural – and tantalizingly sexy – candidate that comes to mind is Suffridge's remarkable family of univalent polynomials, which we discuss in some detail (only to reject it shortly afterwards). The Suffridge polynomials are known to approximate and mimic the Koebe function $\mathcal{K}$ in several ways, so we find it quite surprising that this time they fall short of being extremal. Then we come up with another – newborn – collection of polynomials which, we strongly believe, is the right candidate for the job. Finally, the solution we give for $N = 3$ serves to corroborate the conjecture, and also allows us to compare the extremal properties of the two competing families of polynomials.

2. The Suffridge polynomials – a rejected candidate

In [13], Suffridge introduced an important family of polynomials, which turned out to enjoy a number of elegant extremal properties. Namely, for $N = 1, 2, \dots,$ he defined

$$q_N(z) := \sum_{k=1}^{N} A_{k,N}\, z^k,$$

where

$$A_{k,N} := \frac{N-k+1}{N} \cdot \frac{\sin \pi k/(N+1)}{\sin \pi/(N+1)}, \tag{2.1}$$

and verified that each q_N is univalent in $\mathbb{D}$. Since $A_{1,N} = 1$, we have $q_N \in \mathcal{U}_{N,\mathbb{R}}$ for every N. We further note that $A_{N,N} = 1/N$, which already reflects a certain extremal property of q_N. (Indeed, the highest coefficient a_N of a univalent polynomial (1.4) must satisfy $|a_N| \le 1/N$. To see why, look at the constant term of the monic polynomial $(Na_N)^{-1}p'(z)$ which has no zeros in $\mathbb{D}$.) Moreover, Suffridge showed that whenever $p \in \mathcal{U}_{N,\mathbb{R}}$ is a polynomial of the form (1.4) with $|a_N| = 1/N$, the remaining coefficients of p are also dominated by those of q_N, so that

$$|a_k| \le A_{k,N} \quad \text{for} \quad k = 2, \dots, N. \tag{2.2}$$

When $N \le 4$, it is actually true that every polynomial (1.4) lying in $\mathcal{U}_{N,\mathbb{R}}$ obeys Suffridge's estimates (2.2) unrestrictedly, the assumption that $|a_N| = 1/N$ being no longer needed. (In the nontrivial cases $N = 3$ and $N = 4$, this follows from results of [1, 2, 10] and of [11], respectively.) It is also noteworthy that the polynomial

$$q_3(z) = z + \frac{2\sqrt{2}}{3} z^2 + \frac{1}{3} z^3 \tag{2.3}$$

maximizes $|a_2|$ and $|a_3|$ among all p's of the form $p(z) = z + a_2 z + a_3 z^3$ in $\mathcal{U}_3$, not just in $\mathcal{U}_{3,\mathbb{R}}$; see [1] or [2].

These extremal properties of the Suffridge polynomial q_N seem to indicate that its role in $\mathcal{U}_{N,\mathbb{R}}$ is similar to that of the Koebe function $\mathcal{K}$ in S or $S_{\mathbb{R}}$, the analogy being especially clear-cut for small degrees. On the other hand, for every fixed k, the coefficients $A_{k,N}$ increase to k (i.e., to the kth coefficient of the Koebe function) as $N \to \infty$. Consequently, $\lim_{N\to\infty} q_N = \mathcal{K}$ uniformly on compact subsets of $\mathbb{D}$.

Now let us try and estimate the distance in (1.5) for $p = q_N$. The Koebe-type behavior of q_N suggests, in conjunction with (1.3), that we begin by looking at $|q_N(-1)|$. In fact, Theorem 1.1 tells us that

$$\frac{1}{4} \le \min\{|q_N(\zeta)| : \zeta \in \mathbb{T}\} \le |q_N(-1)|, \tag{2.4}$$

whereas a straightforward computation yields

$$q_N(-1) = -\frac{N+1}{4N}\left[\cos\frac{\pi}{2(N+1)}\right]^{-2},$$

so that $\lim_{N\to\infty} q_N(-1) = -\frac{1}{4}$. We see that the upper bound in (2.4) tends to $\frac{1}{4}$ as $N \to \infty$, meaning that the q_N's are *asymptotically sharp* in the polynomial Koebe problem. Are they also sharp for each individual N?

This last question was explicitly raised in [5], where polynomial analogues of Theorem 1.1 were also touched upon, and this has largely spurred our interest in the problem. While the above discussion seems to provide evidence in favor of a "yes" answer, we now disprove the conjecture (at least in the $\mathcal{U}_{N,\mathbb{R}}$ setting) by showing that the actual answer is a resounding "no," already for $N = 3$. As a matter of fact, the Suffridge polynomial (2.3) fails to be extremal for the Koebe problem in $\mathcal{U}_{3,\mathbb{R}}$ since it loses the game to

$$p_3(z) := z + \frac{2}{\sqrt{5}} z^2 + \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right) z^3, \tag{2.5}$$

another remarkable polynomial from $\mathcal{U}_{3,\mathbb{R}}$, which turns out to be unbeatable. Specifically, the corresponding values of the distance in (1.5) happen to be

$$\min_{\zeta\in\mathbb{T}} |p_3(\zeta)| = |p_3(-1)| = \frac{3-\sqrt{5}}{2} = 0.3819\ldots \tag{2.6}$$

and

$$\min_{\zeta\in\mathbb{T}} |q_3(\zeta)| = \left| q_3\left(-\frac{2\sqrt{2}}{3} \pm \frac{i}{3} \right) \right| = \frac{2}{3\sqrt{3}} = 0.3849\ldots, \tag{2.7}$$

so p_3 does indeed slightly better.

The facts just mentioned (i.e., the univalence of p_3, its extremality in the Koebe problem for $\mathcal{U}_{3,\mathbb{R}}$, and hence also its supremacy over q_3) will be verified in Sections 4 and 5 below. The calculations leading to (2.6) and (2.7) will be provided there as well. But first we have to place the polynomial p_3 where it belongs. Namely, it should be viewed as a member of a certain lordly family, $\{p_N\}$, which we now describe.

3. A new, more promising, family of polynomials and the conjectured solution

Recall, to begin with, that the Chebyshev polynomials of the second kind, U_n, are defined for $n = 0, 1, 2, \ldots$ by the identity

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad \theta \in (-\pi, \pi].$$

Thus,

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t,$$

and so forth. Next, for a positive integer N, we put

$$c_N := \cos\frac{\pi}{N+2}$$

and consider the numbers

$$B_{k,N} := \frac{U'_{N-k+1}(c_N)}{U'_N(c_N)} \cdot U_{k-1}(c_N) \tag{3.1}$$

with $k = 1, \ldots, N$. Finally, we define

$$p_N(z) = \sum_{k=1}^{N} B_{k,N}\, z^k \qquad (N = 1, 2, \ldots).$$

It should be noted that $B_{1,N} = 1$ for each N. Also, rewriting the expression (2.1) for the Suffridge coefficients as

$$A_{k,N} := \frac{N-k+1}{N} \cdot U_{k-1}(c_{N-1}),$$

one might observe a certain – perhaps remote – kinship between the $A_{k,N}$ and the $B_{k,N}$, or equivalently, between the two families of polynomials. The new formulas (3.1) look somewhat more bizarre, if not a bit scary, but there are reasons for them being what they are.

In fact, the polynomials p_N arose quite recently (see [6]) in connection with another extremal problem, which is fairly close in spirit to the current one. The problem was: Given $N \in \mathbb{N}$, maximize the quantity

$$\mu(p) := \min \{\operatorname{Re} p(\zeta) : \zeta \in \mathbb{T}, \operatorname{Im} p(\zeta) = 0\} \tag{3.2}$$

over all polynomials p of the form (1.4) with real coefficients (but without assuming univalence). It was then shown in [6] that the unique maximizing polynomial is precisely p_N, so that the best upper bound for $\mu(p)$ is $\mu(p_N)$, which in turn equals $-1/(4c_N^2)$.

The first two polynomials in the p_N family are

$$p_1(z)\,(= q_1(z)) = z$$

and

$$p_2(z)\,(= q_2(z)) = z + \frac{1}{2}z^2,$$

both being obviously univalent in $\mathbb{D}$. The next one, p_3, is our old friend (2.5) which is again univalent in $\mathbb{D}$, as we shall see in Section 4 below. Then comes

$$p_4(z) = z + \frac{7}{6}z^2 + \frac{2}{3}z^3 + \frac{1}{6}z^4,$$

a polynomial whose univalence has also been established; a nice proof can be found in [4]. In fact, p_4 is even known to be starlike (meaning that it maps $\mathbb{D}$ conformally onto a starlike domain), since it meets the starlikeness criterion given in [9, pp. 515–516]. We have been able to verify univalence for p_5 and p_6 as well, but the case of bigger N's remains open. We do believe that p_N is actually univalent in $\mathbb{D}$, and hence $p_N \in \mathcal{U}_{N,\mathbb{R}}$, for all N. Numerical simulations reinforce this belief substantially.

We further conjecture that the p_N's are extremal in the polynomial Koebe problem, so that for every fixed N in $\mathbb{N}$, p_N minimizes the distance in (1.5) among all $p \in \mathcal{U}_{N,\mathbb{R}}$. The Koebe radius $\rho(\mathcal{U}_{N,\mathbb{R}})$ must then agree with $\min_{\zeta\in\mathbb{T}} |p_N(\zeta)|$, and it is very likely that this last quantity always equals $|p_N(-1)|$, which in turn simplifies to $1/(4c_N^2)$. We go on to claim that the same result should hold in the case of complex coefficients, so that $\mathcal{U}_N$ has presumably the same Koebe radius and the same extremal polynomials as $\mathcal{U}_{N,\mathbb{R}}$. Thus, in particular, it is conjectured that

$$\rho\left(\mathcal{U}_N\right) = \rho\left(\mathcal{U}_{N,\mathbb{R}}\right) = \frac{1}{4\cos^2(\pi/(N+2))}, \qquad N \in \mathbb{N}. \tag{3.3}$$

There are several sources for our certainty, beyond a reasonable doubt, that the conjectured solution is correct. These include the appearance of the p_N polynomials in the cognate extremal problem involving (3.2), as mentioned above, plus the analysis of the case $N = 3$ (which is the bifurcation point between the p_N's and q_N's) to be carried out below, plus the numerical experiments we performed when playing around with polynomials of higher degrees.

4. Polynomials of degree 3: preliminaries

Which functions p of the form

$$p(z) = z + a_2z^2 + a_3z^3 \tag{4.1}$$

are univalent in $\mathbb{D}$? The answer was first obtained in [10] and then rediscovered, via different approaches, in [1] and [2]. To state it (which we only do for the case where a_2 and a_3 are real), we begin by describing the boundary Γ of the univalence region in the (a_2, a_3) plane.

The portion Γ_+ of Γ that lies in the half-plane $\{a_2 \ge 0\}$ can be written as $\Gamma_+ = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where

- γ_1 is the segment of the line $2a_2 - 3a_3 = 1$ with endpoints $\left(0, -\frac{1}{3}\right)$ and $\left(\frac{4}{5}, \frac{1}{5}\right)$;
- γ_2 is the (shorter) arc of the ellipse $a_2^2 + 4(a_3 - \frac{1}{2})^2 = 1$ with endpoints $\left(\frac{4}{5}, \frac{1}{5}\right)$ and $\left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$;
- γ_3 is the segment of the line $a_3 = \frac{1}{3}$ with endpoints $\left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$ and $\left(0, \frac{1}{3}\right)$.

(The two line segments and the arc are assumed to be closed.) We then define

$$\Gamma_- := \left\{(a_2, a_3) \in \mathbb{R}^2 : (-a_2, a_3) \in \Gamma_+\right\}$$

and $\Gamma := \Gamma_+ \cup \Gamma_-$. Thus, Γ is a simple closed curve which is symmetric with respect to the a_3 axis. Finally, we write Ω for the bounded connected component of $\mathbb{C} \setminus \Gamma$ and put $V := \Omega \cup \Gamma$.

The required result from (any of) [1, 2, 10] can now be stated as follows.

Lemma 4.1. *For* $(a_2, a_3) \in \mathbb{R}^2$*, the polynomial* (4.1) *is univalent in* $\mathbb{D}$ *if and only if* (a_2, a_3) *belongs to* V*.*

Obviously enough, the univalence region V is also symmetric with respect to the a_3 axis (as is Γ). This is due to the fact that the polynomial (4.1) and its reflection

$$p^*(z) := -p(-z) = z - a_2z^2 + a_3z^3$$

are, or are not, univalent simultaneously.

The Suffridge polynomial (2.3) corresponds to the vertex $\left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right) = \gamma_2 \cap \gamma_3$ which sticks out in both coordinate directions and has a special, remarkably extreme position in V (along with the symmetric point $\left(-\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$ that represents q_3^*). Thus, when faced with an extremal problem for $\mathcal{U}_{3,\mathbb{R}}$, one is indeed tempted to contemplate q_3 and q_3^* as the most likely extremizers. In our case, however, the actual winners turn out to be p_3 (as defined by (2.5) above) and p_3^*, a fact we shall soon verify. The corresponding points in the coefficient plane are

$$\left(\pm\frac{2}{\sqrt{5}}, \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right); \tag{4.2}$$

both belong to Γ (in fact, the one with the + sign lies on the arc γ_2), so univalence is ensured by Lemma 4.1. The points (4.2) do not appear to enjoy a particularly privileged position, though, so the extremal nature of p_3 and p_3^* can scarcely be viewed as predictable.

Another preliminary question we need to discuss is this: Given a polynomial (4.1) with $a_2, a_3 \in \mathbb{R}$, where does it attain its minimum modulus value on the unit circle $\mathbb{T}$?

We are only interested in the case $a_3 \neq 0$. For a point $z = x + iy \in \mathbb{T}$, a straightforward calculation yields

$$\begin{aligned} |p(z)|^2 &= (1 + a_2 z + a_3 z^2)(1 + a_2 \overline{z} + a_3 \overline{z}^2) \\ &= 1 + a_2^2 + a_3^2 - 2a_3 + 2a_2(1 + a_3)x + 4a_3 x^2 =: \Phi(x). \end{aligned} \tag{4.3}$$

Originally, $x := \operatorname{Re} z$ runs through the interval $[-1, 1]$, but we extend the quadratic polynomial $\Phi(x)$ to all $x \in \mathbb{R}$. Then

$$\Phi'(x) = 2a_2(1 + a_3) + 8a_3 x,$$

and the only zero of this derivative is

$$x_0 = -\frac{a_2(1 + a_3)}{4a_3}. \tag{4.4}$$

The function $\Phi(x)$ therefore attains its minimum (if $a_3 > 0$) or maximum (if $a_3 < 0$) at x_0, its value at the critical point being

$$\Phi(x_0) = (1 - a_3)^2 \left(1 - \frac{a_2^2}{4a_3}\right), \tag{4.5}$$

as verified by direct computation. In particular, this last quantity will be nonnegative whenever x_0 happens to be in $[-1, 1]$; to see why, recall (4.3).

In terms of p, two types of behavior may occur. To distinguish between them, we now introduce the appropriate terminology.

Definition 4.2. *A polynomial p is said to be of type I if*

$$\min\{|p(z)| : z \in \mathbb{T}\} = \min\{|p(-1)|, |p(1)|\}. \tag{4.6}$$

Otherwise we say that p is of type II.

It should be noted that, for a polynomial p with nonnegative coefficients, (4.6) simplifies to

$$\min\{|p(z)| : z \in \mathbb{T}\} = |p(-1)|. \tag{4.7}$$

Thus, polynomials of type I are essentially those that mimic the Koebe function $\mathcal{K}$ by sharing its property (1.3).

The above discussion leads us to the following conclusion.

Lemma 4.3. *Let p be a polynomial of the form* (4.1) *with real coefficients and with $a_3 \neq 0$. In order that p be of type II, it is necessary and sufficient that $a_3 > 0$ and*

$-1 < x_0 < 1$, *where* x_0 *is defined by* (4.4). *In this case,*

$$\min\{|p(z)| : z \in \mathbb{T}\} = |p(x_0 \pm iy_0)| = |1 - a_3| \left(1 - \frac{a_2^2}{4a_3}\right)^{1/2}, \tag{4.8}$$

where $y_0 := \sqrt{1 - x_0^2}$. *Moreover,* $x_0 \pm iy_0$ *are then the only points of* $\mathbb{T}$ *where the minimum in question is attained.*

This result allows us to compute the quantity (1.5) for $p = p_3$ and $p = q_3$, thus verifying the announced formulas (2.6) and (2.7).

The polynomial p_3 has $a_2 = \frac{2}{\sqrt{5}}$ and $a_3 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$, and plugging this into (4.4) gives

$$x_0 = -\frac{1}{4} - \frac{7}{20}\sqrt{5} = -1.0326\ldots.$$

It now follows from Lemma 4.3 that p_3 is of type I, and so

$$\min\{|p_3(z)| : z \in \mathbb{T}\} = |p_3(-1)|. \tag{4.9}$$

The right-hand side of (4.9) reduces to $(3 - \sqrt{5})/2$, and we arrive at (2.6).

As to the Suffridge polynomial q_3, this time we have

$$x_0 = -\frac{2\sqrt{2}}{3}(= -a_2) = -0.9428\ldots,$$

so Lemma 4.3 tells us that q_3 is of type II. The corresponding y_0 equals $\frac{1}{3}(= a_3)$, and substituting the appropriate values into (4.8) yields (2.7).

5. Polynomials of degree 3: solution

For a polynomial p, we put

$$m(p) := \min\{|p(\zeta)| : \zeta \in \mathbb{T}\}.$$

To solve the Koebe problem for $\mathcal{U}_{3,\mathbb{R}}$, we need to minimize the functional $m(p)$ over all $p \in \mathcal{U}_{3,\mathbb{R}}$. This is done in Theorem 5.1 below, where the minimizing polynomials are exhibited; as promised, these are shown to be p_3 and p_3^*.

A couple of conventions will be made. First, if X is a class of polynomials and $F \in X$, we say that F *is extremal for* X to mean that

$$\inf\{m(p) : p \in X\} = m(F).$$

Secondly, every polynomial p in $\mathcal{U}_{3,\mathbb{R}}$ will be identified, via (4.1), with the corresponding point (a_2, a_3) in the plane (or rather in the univalence region V coming from Lemma 4.1); we shall occasionally write $p = (a_2, a_3)$ to make this explicit. Also, given a set $M \subset V (\subset \mathbb{R}^2)$, we may now use the notation $p \in M$ without any risk of confusion.

Theorem 5.1. *The only extremal polynomials for the class* $\mathcal{U}_{3,\mathbb{R}}$ *are* p_3, *as defined by* (2.5), *and its reflection* p_3^*.

As a consequence, we see that the Koebe radius for $\mathcal{U}_{3,\mathbb{R}}$ equals $m(p_3)$, which agrees with

$$\frac{1}{4\cos^2\frac{\pi}{5}}=\frac{3-\sqrt{5}}{2},$$

the conjectured (and now established) value of $\rho(\mathcal{U}_{3,\mathbb{R}})$ from (3.3). It only remains to prove Theorem 5.1.

Proof. The extremal polynomials must live on the boundary, Γ, of the univalence region V. By symmetry, it suffices to consider

$$\Gamma_+=\{(a_2,a_3)\in\Gamma:\,a_2\ge 0\},$$

which in turn decomposes as $\gamma_1\cup\gamma_2\cup\gamma_3$; see the preceding section for definitions.

We begin by looking at the values of $m(p)$ when $p\in\gamma_1$. It is easy to check that, whenever $p=(a_2,a_3)$ is a point of γ_1 with $a_3>0$, we have

$$\frac{a_2(1+a_3)}{4a_3}\ge 1.$$

Equivalently, the number x_0 given by (4.4) is in $(-\infty,-1]$ for any such point, and we deduce from Lemma 4.3 that the polynomials belonging to γ_1 are all of type I. It is also clear that every polynomial $p\in\gamma_1$ satisfies

$$|p(-1)|=1-a_2+a_3\le 1+a_2+a_3=|p(1)|. \tag{5.1}$$

Furthermore, on γ_1 we have $a_3=\frac{1}{3}(2a_2-1)$, whence

$$1-a_2+a_3=\frac{1}{3}(2-a_2), \tag{5.2}$$

the permissible values of a_2 being those in $[0,\frac{4}{5}]$. We now combine (4.6), (5.1), and (5.2) to get

$$m(p)=|p(-1)|=\frac{1}{3}(2-a_2)$$

for each polynomial $p=(a_2,a_3)\in\gamma_1$. Consequently,

$$\inf\{m(p):\,p\in\gamma_1\}=\frac{1}{3}\left(2-\frac{4}{5}\right)=\frac{2}{5}=0.4. \tag{5.3}$$

Next, we turn to the case where $p=(a_2,a_3)\in\gamma_2$. We know that γ_2 contains polynomials of both types, and we write σ_{I} (resp., σ_{II}) for the set of all polynomials of type I (resp., II) that are in γ_2. In fact, σ_{I} and σ_{II} are disjoint subarcs of γ_2 whose common endpoint $\widetilde{p}=(\widetilde{a}_2,\widetilde{a}_3)\in\gamma_2$ is determined by the relation

$$\widetilde{a}_2=\frac{4\widetilde{a}_3}{1+\widetilde{a}_3}.$$

(Thus, plugging $a_2=\widetilde{a}_2$ and $a_3=\widetilde{a}_3$ into (4.4) yields $x_0=-1$. A bit of inspection shows that γ_2 contains exactly one point with this property. We note that the number $\widetilde{a}_3$ satisfies $\frac{1}{5}<\widetilde{a}_3<\frac{1}{3}$ and coincides with the unique positive root of the equation $t^3+t^2+3t-1=0$.) Precisely speaking, σ_{I} is the closed subarc of γ_2 with endpoints $\left(\frac{4}{5}, \frac{1}{5}\right)$ and $(\widetilde{a}_2, \widetilde{a}_3)$, while $\sigma_{\mathrm{II}} = \gamma_2 \setminus \sigma_{\mathrm{I}}$ is the complementary (half-open) subarc with endpoints $(\widetilde{a}_2, \widetilde{a}_3)$ and $\left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$.

Rewriting the equation of the ellipse (of which γ_2 forms part) as

$$a_2^2 = 4a_3(1 - a_3), \tag{5.4}$$

we may parametrize γ_2 in the form

$$\gamma_2 = \left\{ \left(2\sqrt{t(1-t)},\, t\right) : t \in \left[\frac{1}{5}, \frac{1}{3}\right] \right\},$$

where σ_{I} and σ_{II} correspond to the parameter ranges $\left[\frac{1}{5}, \widetilde{a}_3\right] =: J_{\mathrm{I}}$ and $\left(\widetilde{a}_3, \frac{1}{3}\right] =: J_{\mathrm{II}}$.

Accordingly, the quantity $m(p)$ with $p \in \gamma_2$ admits a fairly simple expression in terms of the a_3 coordinate. Namely, for $p = (a_2, a_3) \in \sigma_{\mathrm{I}}$, we combine (4.7) with (5.4) to obtain

$$m(p) = |p(-1)| = 1 - a_2 + a_3 = 1 - 2\sqrt{a_3(1-a_3)} + a_3, \tag{5.5}$$

whereas for $p = (a_2, a_3) \in \sigma_{\mathrm{II}}$ we invoke (4.8) in conjunction with (5.4) to get

$$m(p) = (1 - a_3)\left(1 - \frac{a_2^2}{4a_3}\right)^{1/2} = \sqrt{a_3}(1 - a_3). \tag{5.6}$$

Differentiating, we find that the function

$$\varphi(t) := 1 - 2\sqrt{t(1-t)} + t, \qquad t \in J_{\mathrm{I}},$$

has a minimum at

$$t_* := \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$$

(which is an interior point of J_{I}) and

$$\min\{\varphi(t) : t \in J_{\mathrm{I}}\} = \varphi(t_*) = \frac{3 - \sqrt{5}}{2}.$$

Moreover, t_* is the only point in J_{I} where the minimum in question is attained. We further observe that the function

$$\psi(t) := \sqrt{t}(1 - t), \qquad t \in J_{\mathrm{II}},$$

is increasing on its domain $J_{\mathrm{II}}(\subset (0, \frac{1}{3}])$; this is again verified by differentiation.

From (5.5) and (5.6) we know that $m(p)$ equals $\varphi(a_3)$ when $p \in \sigma_{\mathrm{I}}$, and $\psi(a_3)$ when $p \in \sigma_{\mathrm{II}}$. The critical value t_* of the a_3 variable corresponds to the point $(2/\sqrt{5}, t_*) \in \sigma_{\mathrm{I}}$, which represents the polynomial p_3. Because $m(p)$ is also continuous at $\widetilde{p}$ (as it is everywhere else), the above facts about the φ and ψ functions allow us to conclude that

$$\inf\{m(p) : p \in \gamma_2\} = m(p_3) = \frac{3 - \sqrt{5}}{2} \tag{5.7}$$

and that p_3 is the only extremal polynomial for γ_2.

It remains to consider the case where $p = (a_2, a_3) \in \gamma_3$. Since $0 \le a_2 \le \frac{2\sqrt{2}}{3}$ and $a_3 = \frac{1}{3}$, the formula (4.4) yields $x_0 = -a_2$, whence in particular

$$-1 < -\frac{2\sqrt{2}}{3} \le x_0 \le 0.$$

It now follows from Lemma 4.3 that p is of type II, and moreover,

$$m(p) = \frac{1}{3}\left(4 - 3a_2^2\right)^{1/2}.$$

Clearly, this is minimized by assigning the largest admissible value, $\frac{2\sqrt{2}}{3}$, to the a_2 variable. In other words, the only extremal polynomial for γ_3 is $q_3 = \left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$, and so

$$\inf\{m(p) : p \in \gamma_3\} = m(q_3) = \frac{2}{3\sqrt{3}}. \tag{5.8}$$

Finally, a quick glance at (5.3), (5.7) and (5.8) reveals that the smallest of the three infima is that over γ_2. This implies the extremality (and uniqueness) of p_3 among all polynomials $p = (a_2, a_3) \in \mathcal{U}_{3,\mathbb{R}}$ with $a_2 \ge 0$. By symmetry, a similar role is played by p_3^* among the $\mathcal{U}_{3,\mathbb{R}}$ polynomials with $a_2 < 0$. The proof is complete. □

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