### AVERAGE PREDICTIVE CONTROL FOR NONLINEAR DISCRETE DYNAMICAL SYSTEMS

#### D. DMITRISHIN, I.E. IACOB, AND A. STOKOLOS

ABSTRACT. We explore the problem of stabilization of unstable periodic orbits in discrete nonlinear dynamical systems. This work proposes the generalization of predictive control method for resolving the stabilization problem. Our method embodies the development of control method proposed by B.T. Polyak. The control we propose uses a linear (convex) combination of iterated functions. With the proposed method auxiliary, the problem of robust cycle stabilization for various cases of its multipliers localization is solved. An algorithm for finding a given length cycle when its multipliers are known is described as a particular case of our method application. Also, we present numerical simulation results for some well-known mappings and the possibility of further generalization of this method.

#### 1. INTRODUCTION

Nonlinear dynamical systems are often characterized by extremely unstable movements in the phase space, defined as chaotic movements [5]. In practice, it is generally desirable to suppress or prevent such chaotic behavior due to its adverse effect on the physical systems normal operation. Due to its theoretical significance and engineering applicability, much attention has been paid to the problem of chaos controlling in various fields and numerous studies [2, 3, 12, 19]. By chaos control we mean small external influences on the system or a small change in the system structure in order to transform the system chaotic behavior into a regular (or chaotic, still specific with other properties) one [11].

It is assumed that the dynamical system includes a chaotic attractor, which contains a countable set of unstable cycles with different periods. If, by using some control effect, a certain cycle is stabilized locally, the system path will remain in its neighborhood, i.e. regular movements will be observed in the system. Hence, one of the ways for chaos controlling refers to the local stabilization of certain orbits from a chaotic attractor.

The problem of stabilizing cycles is closely related to the problem of finding periodic points. The various control schemes [1,4,9] that were proposed for solving these problems can be divided into two large groups: direct and indirect methods. The indirect methods either use the initial mapping T iterations or imply the construction a system of which order is T times greater than the initial system order (T being the desired cycle length). Then one of the methods of finding a fixed point is applied. The most common among fixed point finding techniques is the Newton-Raphson relaxation method and its further modifications [10,16,19,28]. The next step is to select periodic points from the entire set of fixed points. In direct methods, all the points in the cycle are found concurrently, i.e. the whole cycle is stabilized. In this case, the initial system is closed by control, based on the

<sup>2010</sup> Mathematics Subject Classification. Primary: 37F99; Secondary: 34H10.

Key words and phrases. Non-linear discrete systems, stabilization, predictive control.

feedback principle [7,17,22,27]. Among such control schemes, the most simple in terms of physical implementation are the linear ones. However, they have significant limitations, as they can only be applied to a narrow domain of the space of parameters that are part of the initial nonlinear system [6,25,29]. To overcome the restricted stabilization condition Ushio and Yamamoto [26] introduced a prediction-based feedback control method for discrete chaotic systems with accurate mathematical model. B.T. Polyak [20] (see also [21]) proposed a direct predictive control method that works well for one-dimensional as well as multidimensional maps. L. Shalby [24] used predictive feedback control method for stabilization of continuous time systems.

Another possible control schemes classification into two groups: methods using the Jacobi matrix and methods not based on this matrix. Naturally, it is assumed that the Jacobi matrix at the cycle points is not properly known, otherwise it would be possible to use the whole powerful apparatus of the linear control theory applied to systems linearized in the cycle neighborhood. The Jacobi matrix is an indispensable attribute of Newton-Raphson-type methods. This matrix is also used in one of the modifications derived from Polyak predictive control method.

One of the main disadvantages of Polyak scheme refers to the need for knowing the Jacobi matrix for the cycle, or at least the need for sufficiently good estimates of the cycle multipliers. The research exposed herein is purposed to improve the Polyak method by replacing it with mixed predicted values. A Jacobi matrix representation for the *T* cycle at a controlled system is found through the Jacobi matrix of the same cycle in the initial system. Therefore, the correspondence between the cycle multipliers of the open loop system and those of closed loop system is established.

Below, it is assumed that the initial system cycle multipliers are not exactly known, we know only know the range of their localization. Then the solution of cycles robust stabilization problem for various localization of multipliers is given, and taking into account these general provisions the Polyak method is considered. It is worth noting that in the general case of complex multipliers, we must know precisely enough their localization regions. At the end, the applications of proposed predictive control scheme to stabilize the cycles of some common systems in Physics literature are considered.

#### 2. PROBLEM STATEMENT.

Considered is a nonlinear discrete system

(1) 
$$x_{n+1} = f(x_n), x_n \in \mathbb{R}^m, n = 1, 2, ...,$$

where f(x) is a differentiable vector function of corresponding dimension. It is assumed that the system (1) has an invariant convex set A, i.e. if  $\xi \in A$ , then  $f(\xi) \in A$ . We emphasize that we do not assume that the set A is a minimally convex set. It is also assumed that this system has one or several unstable T cycles  $\{\eta_1, \ldots, \eta_T\}$ , where all vectors are different and belong to the invariant set A, i.e.  $\eta_{j+1} = f(\eta_j), j = 1, \ldots, T - 1, \eta_1 = f(\eta_T)$ . The considered unstable cycles multipliers are determined as eigenvalues of the product of Jacobi matrices  $\prod_{j=1}^{T} f'(\eta_{T-j+1})$  of dimensions  $m \times m$  at the cycle points. The matrix  $\prod_{i=1}^{T} f'(\eta_{T-j+1})$  is called the Jacobi matrix of the cycle  $\{\eta_1, \ldots, \eta_T\}$ . Typically, a priori the

cycles of the system (1) are not known. Consequently, the spectrum  $\{\mu_1, \ldots, \mu_m\}$  of the matrix  $\prod_{j=1}^{T} f'(\eta_{T-j+1})$  is unknown as well. The spectrum elements are called cycle multipliers. Below, we assume that some estimates on the localization set *M* for the cycle multipliers are known.

Let us consider the control system

$$(2) x_{n+1} = F(x_n),$$

where 
$$F(x) = \sum_{j=1}^{N} \theta_j f^{((j-1)T+1)}(x), f^{(1)}(x) = f(x), f^{(k)}(x) = f\left(f^{(k-1)}(x)\right), k = 2, \dots, T.$$

The numbers  $\theta_1, \ldots, \theta_N$  are real. It can be easily verified that at  $\sum_{j=1}^N \theta_j = 1$  the system (2)

also includes the cycle  $\{\eta_1, \ldots, \eta_T\}$ . We aim to choose such parameter *N* and coefficients  $\{\theta_1, \ldots, \theta_N\}$  so that the system (2) cycle  $\{\eta_1, \ldots, \eta_T\}$  would be locally asymptotically stable. Naturally, when constructing these coefficients, there will be used information on set *M* of multipliers localization. It is also desirable [13, 14] to fulfill an additional condition: the system (1) invariant convex set *A* must be also invariant for the system (2). This requirement will be fulfilled, for example, if  $0 \le \theta_j \le 1, j = 1, \ldots, N$ .

Polyak method [20] utilizes the case  $\theta_1 = 1$ ,  $\theta_2 = \cdots = \theta_{N-2} = 0$ ,  $\theta_{N-1} = -\theta_N = \varepsilon$ . Regarding the set M, it was assumed that  $M = \mathbb{D} \cup \{\mu^*\}$  where  $\mathbb{D} = \{z : |z| < 1\}$  is the central unit circle on the complex plane, and  $\mu^*$  is a known real number. In the case m = 1 the required coefficient formula has the form  $\varepsilon = \frac{1 \mp (|\mu^*| / \rho)^{-\frac{1}{T}}}{(\mu^*)^{N-2} (\mu^* - 1)}$  where  $0 < \rho < 1$ . In this article the control problem is solved for a wider class of multipliers localization set M.

### 3. CONSTRUCTING THE JACOBI MATRIX FOR A CONTROLLED SYSTEM

Investigating stability of *T* cycles of the system (2) consists in constructing of Jacobi matrix  $\prod_{j=1}^{T} F'(\eta_{T-j+1})$  of that cycle and studying the eigenvalues of this matrix. To derive the Jacobi matrix, we use the ideas from [20].

Let  $J_j = f'(\eta_j)$ , j = 1, ..., T, then we write the Jacobi matrix of the system (1) cycle  $\{\eta_1, ..., \eta_T\}$  as  $J = J_T \cdot ... \cdot J_1$ . We introduce the following auxiliary matrices:

$$A_1 = I, A_2 = J_1, A_3 = J_2 \cdot J_1, \dots, A_{T-1} = J_{T-1} \cdot \dots \cdot J_1$$

 $(I - unity matrix of order m \times m);$ 

$$B_1 = J_T \cdot \ldots \cdot J_1 = J, \ B_2 = J_T \cdot \ldots \cdot J_2, \ \ldots, \ B_T = J_T;$$

then  $B_k A_k = J$ , k = 1, ..., T,  $A_k B_k = (J_{k-1} \cdot ... \cdot J_1) \cdot (J_T \cdot ... \cdot J_k)$  and, consequently,  $(A_k B_k)^s = A_k J^{s-1} B_k$ , s = 1, 2, ...

By chain rule:

$$\left(f^{(s)}(x)\right)'\Big|_{x=\eta_i} = \left(f^{(s-1)}(x)\right)'\Big|_{x=\eta_{i+1}} \cdot (f(x))'\Big|_{x=\eta_i} = \left(f^{(s-1)}(x)\right)'\Big|_{x=\eta_{i+1}} \cdot J_i,$$

we get

$$\left(f^{((j-1)T)}(x)\right)'\Big|_{x=\eta_i} = A_i J^{j-2} B_i, \ j=2,\ldots,N$$

and therefore

$$\left(f^{((j-1)T+1)}(x)\right)'\Big|_{x=\eta_i} = J_i A_i J^{j-2} B_i, \ j=2,\ldots,N.$$

Next we find that

$$F'(\eta_i) = \sum_{j=1}^N \theta_j \left( f^{((j-1)T+1)}(x) \right)' \Big|_{x=\eta_i} = \theta_1 J_i + \sum_{j=2}^N \theta_j J_i A_i J^{j-2} B_i.$$

For the Jacobi matrix of the system (2) cycle  $\{\eta_1, \ldots, \eta_T\}$  we can write:

$$F'(\eta_T) \cdot \ldots \cdot F'(\eta_1) = J_T \left( \theta_1 I + A_T \left( \sum_{j=2}^N \theta_j J^{j-2} \right) B_T \right) \cdot J_{T-1} \left( \theta_1 I + A_{T-1} \left( \sum_{j=2}^N \theta_j J^{j-2} \right) B_{T-1} \right) \cdot \ldots \cdot J_1 \left( \theta_1 I + A_1 \left( \sum_{j=2}^N \theta_j J^{j-2} \right) B_1 \right)$$

Taking into account that

$$J_k \left( \theta_1 I + A_k \left( \sum_{j=2}^N \theta_j J^{j-2} \right) B_k \right) =$$

$$J_k A_k \left( \theta_1 I + \left( \sum_{j=2}^N \theta_j J^{j-2} \right) B_k A_k \right) A_k^{-1} = J_k A_k \left( \sum_{j=1}^N \theta_j J^{j-1} \right) A_k^{-1}$$

$$A_k \quad \text{it follows that}$$

and  $J_k A_k = A_{k+1}$  it follows that

$$F'(\eta_{T}) \cdot \ldots \cdot F'(\eta_{1}) = \\ = J_{T}A_{T} \left(\sum_{j=1}^{N} \theta_{j} J^{j-1}\right) A_{T}^{-1} \cdot J_{T-1}A_{T-1} \left(\sum_{j=1}^{N} \theta_{j} J^{j-1}\right) A_{T-1}^{-1} \cdot \ldots \cdot \\ \cdot J_{1}A_{1} \left(\sum_{j=1}^{N} \theta_{j} J^{j-1}\right) A_{1}^{-1} = J \left(\sum_{j=1}^{N} \theta_{j} J^{j-1}\right)^{T}$$

(The reader is gently advised that the superscript *T* in the formula above and all subsequent formulas denotes power, not transpose.)

For deriving the Jacobian formula above it was assumed that the matrix *J* was not degenerated. This limitation can be easily circumvented using a well-known topological technique: considering the matrix  $J + \delta I$  instead of the degenerated matrix *J* and after all calculations taking the limit as  $\delta \rightarrow 0$ . Thus, the following result is obtained.

**Lemma 3.1.** The Jacobi matrix of the cycle  $\{\eta_1, \ldots, \eta_T\}$  in the system (2) can be represented as

(3) 
$$J\left(\sum_{j=1}^{N}\theta_{j}J^{j-1}\right)^{T},$$

where *J* is the Jacobi matrix of the cycle  $\{\eta_1, \ldots, \eta_T\}$  in the system (1).

We now consider another control system, instead of system (2):

(4) 
$$x_{n+1} = f\left(\theta_1 x_n + \sum_{j=2}^N \theta_j f^{((j-1)T)}(x_n)\right).$$

When  $\sum_{j=1}^{N} \theta_j = 1$  then the system (4) preserves the cycle  $\{\eta_1, \dots, \eta_T\}$ . In addition, ac-

cording to formula (3), the Jacobi matrix of the system (4) cycle is expressed in the terms of Jacobi matrix of the system (1). The advantage of the control system (4) over the system (2) consists of a fewer calculation of the values for function f(x) (more precisely, the difference is N - 2).

### 4. MAIN RESULT

All results presented in this section are formulated for system (2), however they hold without change for system (4).

**Theorem 4.1.** Suppose  $f \in C^1$  and that the system (1) has an unstable *T* cycle with multipliers  $\{\mu_1, \ldots, \mu_m\}$ . Then this cycle will be a locally asymptotically stable cycle of the system (2) if

$$\mu_j \left[ r \left( \mu_j \right) \right]^T \in \mathbb{D}, \quad j = 1, \dots, m,$$

where  $r(\mu) = \sum_{j=1}^{N} \theta_j \mu^{j-1}$ .

*Proof.* According to Lemma 3.1, the characteristic polynomial for a system of linear approximation in the cycle neighbourhood in the case of system (2) can be written as  $\varphi(\lambda) = \det (\lambda I - J[r(J)]^T)$ . By reducing the matrix *J* to the Jordan form, this characteristic polynomial can be represented as  $\varphi(\lambda) = \prod_{j=1}^{m} (\lambda I - \mu_j [r(\mu_j)]^T)$ , from where the theorem conclusion follows.

Note that the condition r(1) = 1 is obligatory. If, additionally,  $\theta_j \in [0, 1]$  for j = 1, ..., N, then  $\mu_j [r(\mu_j)]^T \in \overline{\mathbb{D}}$  when  $\mu_j \in \overline{\mathbb{D}}$ , and hence  $|\mu_j [r(\mu_j)]^T| < |\mu_j|^{1+T}$ . This means that if some multiplier of the system (1) cycle lies in the unit circle, the corresponding multiplier of the system (2) will lie closer to zero. Thus for the closed loop system, the stabilization quality is improving. Various estimates for multipliers allow us to construct control systems that stabilize cycles.

4.1. **Case** 
$$M = {\mu_1, ..., \mu_m}$$
. If the multipliers are exactly known, we can choose  $N = m+1$  and the coefficients  ${\theta_1, ..., \theta_{m+1}}$  from the condition  $r(\mu) = \sum_{j=1}^{m+1} \theta_j \mu^{j-1} = \frac{1}{\prod_{k=1}^m (1-\mu_k)} \prod_{k=1}^m (\mu - \mu_k)$ .

Then from Theorem 4.1 we get the following conclusion.

*Conclusion.* Suppose that  $f \in C^1$  and the system (1) has an unstable *T* cycle with multipliers  $\{\mu_1, \ldots, \mu_m\}$ , and the coefficients  $\theta_1, \ldots, \theta_{m+1}$  are found as exposed above. Then this cycle will be a locally asymptotically stable cycle of system (2). Moreover, if the initial point belongs to the cycle basin of attraction, the convergence to the cycle is superlinear.

The superlinearity of the convergence rate follows from the fact that all multipliers of system (2) { $\eta_1, ..., \eta_T$ } cycle turn out to be zero.

Note that the authors are unaware about any other method that allow to stabilize a cycle by knowing the cycle multipliers only. Unfortunately, in a typical situation the multipliers are either unknown. The best we can expect is to localize them approximately. What to do in that case is considered in the next section.

4.2. **Case**  $M = \{z : \operatorname{Re} z \le 0\} \cup \mathbb{D}$ .

**Theorem 4.2.** Suppose  $f \in C^1$  and that system (1) has an unstable *T*-cycle with multipliers  $\{\mu_1, \ldots, \mu_m\}$  satisfying the conditions:

$$|\mu_j - \hat{\mu}_j| < \delta_j, \text{ Re } \{\mu_j\} \le 0, \ j = 1, \dots, n_1, \ |\mu_j| < 1, \ j = n_1 + 1, \dots, m.$$

Let the coefficients  $\theta_j$ , j = 1, ..., N, of the system (2) be determined from the condition

$$\sum_{j=1}^{n_1+1} \theta_j \mu^{j-1} = \frac{1}{\prod\limits_{k=1}^{n_1} (1-\widehat{\mu}_k)} \prod\limits_{k=1}^{n_1} (\mu - \widehat{\mu}_k) \quad (here \ N = n_1 + 1).$$

Then, for sufficiently small values  $\delta_j$ ,  $j = 1, ..., n_1$ , the T-cycle will be a locally asymptotically stable cycle of system (2).

*Proof.* Since Re  $\{\mu_j\} < 0, j = 1, ..., n_1$ , then all coefficients  $\theta_j > 0, j = 1, ..., n_1$ . That means that  $|\mu[r(\mu)]^T| < |\mu|^{1+T}$  when  $|\mu| < 1$ , i.e. the eigenvalues of the Jacobi matrix of system (2) cycle corresponding to multipliers  $\mu_j, j = n_1 + 1, ..., m$ , are smaller than the multipliers absolute values. Let  $\delta_j = 0, j = 1, ..., n_1$ , then the eigenvalues corresponding to multipliers  $\mu_j, j = 1, ..., n_1$ , are equal to zero. When  $\delta_j, j = 1, ..., n_1$ , are sufficiently small in magnitude, these eigenvalues will lie in the central unit circle, as follows from Rouche theorem. Thus, all eigenvalues will less than 1 in absolute value, which means local asymptotic stability.

4.3. **Case** 
$$M = \mathbb{D} \cup \{\mu^*\}$$
,  $|\mu^*| > 1$  [20]. In [20], the coefficients  $\theta_1, ..., \theta_N$  were chosen as  $\theta_1 = 1, \theta_2 = ... = \theta_{N-2} = 0, \theta_{N-1} = -\theta_N = \varepsilon$ , where  $\varepsilon = \frac{1 \mp (|\mu^*| / \rho)^{-\frac{1}{T}}}{(\mu^*)^{N-2} (\mu^* - 1)}, 0 < \rho < 1$ .

Such a choice ensures that the multipliers belong to the open central unit interval corresponding to  $\mu^*$ . However, the condition  $\left|\mu\left[r(\mu)\right]^T\right| < 1$  with  $|\mu| < 1$  is not necessarily satisfied. Nevertheless, the value  $\varepsilon$  can be made arbitrarily small by choosing the number *N* large. And then, from Rouche's theorem, it follows that with a sufficiently large *N* the other multipliers will remain within the central unit circle. This ensures the local asymptotic stability of the system (2) cycle.

When it is known that  $\mu^* < -1$ , the control scheme can be simplified, namely:

(5) 
$$x_{n+1} = \theta_1 f(x_n) + \theta_2 f^{(T+1)}(x_n),$$

where  $\theta_1 = \frac{|\mu^*|}{1+|\mu^*|}, \theta_2 = \frac{1}{1+|\mu^*|}.$ 

4.4. **Case**  $M = \mathbb{D} \cup \{\mu^*, \overline{\mu}^*\}, |\mu^*| > 1$ . The case of general localization of multipliers  $\{\mu_1, \ldots, \mu_m\}$  for the system (1) cycle was considered in [20] but only for T = 1. In that case, the coefficients  $\theta_1, \ldots, \theta_N$  were no longer scalars but matrices and, as before, were chosen as  $\theta_1 = I, \theta_2 = \ldots = \theta_{N-2} = 0, \theta_{N-1} = -\theta_N = \varepsilon$ , where *I* is identity matrix, 0 is zero matrix,  $\varepsilon = S\Lambda S^{-1}, \Lambda = \text{diag} \{\varepsilon_1, \ldots, \varepsilon_m\}, \varepsilon_j = \frac{1 + e^{i\varphi} (\rho / |\mu_j|)}{(\mu_j)^{N-2} (\mu_j - 1)}$ , if  $|\mu_j| < 1$  and  $\varepsilon = 0$  if  $|\mu_j| > 1$  and  $0 < \varepsilon < 1$ ,  $\theta \in \{0, \pi\}$  if  $\mu_j$  as a real number. The

1, and  $\varepsilon_j = 0$ , if  $|\mu_j| > 1$ , and  $0 < \rho < 1$ ,  $\varphi \in \{0, \pi\}$  if  $\mu_j$  as a real number. The matrix *S* consists of the eigenvectors of the Jacobi matrix *J* for equilibrium point. Thus, to apply the stabilization method, it is necessary to know not only all the multipliers of the equilibrium, but also the Jacobi matrix itself. That is impossible when the equilibrium is not known.

Now, let us we apply the scheme (2). Let  $\mu^* = \rho e^{i\varphi}$ . Then

$$r(\mu) = \frac{(\mu - \mu^*) (\mu - \overline{\mu}^*)}{(1 - \mu^*) (1 - \overline{\mu}^*)} = \frac{\rho^2}{\rho^2 - 2\rho \cos \varphi + 1} + \frac{-2\rho \cos \varphi}{\rho^2 - 2\rho \cos \varphi + 1} \mu + \frac{1}{\rho^2 - 2\rho \cos \varphi + 1} \mu^2$$

If the complex number  $\mu^*$  lies in the left half-plane, then the coefficients of polynomial  $r(\mu)$  are positive, so  $|\mu[r(\mu)]^T| < |\mu|^{1+T}$  with  $|\mu| < 1$ . Therefore, each multiplier of system (2) cycle lying in the central unit circle turns out to be in absolute value less then the multiplier of the corresponding cycle of the system (1). Also, the multipliers corresponding to  $\mu^*$  and  $\overline{\mu}^*$  change to zero. If the multipliers  $\mu^*$  and  $\overline{\mu}^*$  are not exactly known, but they can be well estimated, then for the coefficients  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$ , although different from the calculated ones, the values of the polynomial with these coefficients at points  $\mu^*$  and  $\overline{\mu}^*$  will not exceed 1 in absolute value, as follows from the Rouche theorem. The desired control system is

$$x_{n+1} = \theta_1 f(x_n) + \theta_2 f^{(T+1)}(x_n) + \theta_3 f^{(2T+1)}(x_n),$$
  
where  $\theta_1 = \frac{\rho^2}{\rho^2 - 2\rho \cos \varphi + 1}, \theta_2 = \frac{-2\rho \cos \varphi}{\rho^2 - 2\rho \cos \varphi + 1}, \theta_3 = \frac{1}{\rho^2 - 2\rho \cos \varphi + 1},$ 

4.5. **Case** T = 1,  $M = \lfloor -\mu^*, \mu^* \rfloor$ ,  $M = \lfloor -\mu^*, 1 \rfloor$ . Suppose  $M = \lfloor -\mu^*, \mu^* \rfloor$ . From Theorem 4.1 it follows that in order to stabilize the equilibrium, it would be necessary to construct a polynomial  $\mu r(\mu)$ , so that r(1) = 1 and  $|\mu r(\mu)| \le 1$  for all  $|\mu| < \mu^*$ .

**Theorem 4.3.** Let  $f \in C^1$  and the system (1) has unstable equilibrium with multipliers  $\{\mu_1, \ldots, \mu_m\} \subset [-\mu^*, \mu^*]$ . Let the value N be odd and be chosen from the condition  $\csc \frac{\pi}{2N} > \mu^*$ , and the coefficients  $\theta_1, \ldots, \theta_N$  from the condition

$$\mu r(\mu) = \mu \sum_{j=1}^{N} \theta_j \mu^{j-1} = (-1)^{\frac{N-1}{2}} T_N\left(\mu \sin \frac{\pi}{2N}\right),$$

where  $T_N(x)$  is the first kind Chebyshev polynomial of odd order N. Then this equilibrium will be a locally asymptotically stable equilibrium of system (2) (modulo a finite number of cases when  $\mu_j = \frac{\cos \pi k/N}{\sin \pi/2N}$  for some k = 1, ..., N - 1). The proof follows from the properties of the first kind Chebyshev polynomials:  $|\mu r(\mu)| \le 1$  at  $\left|\mu \sin \frac{\pi}{2N}\right| \le 1$ , r(1) = 1. Note that  $\mu^* \to \infty$   $(N \to \infty)$  with asymptotics  $\frac{2}{\pi}N$ . Now we will consider the case of  $M = |-\mu^*, 1|$ .

**Theorem 4.4.** Let  $f \in C^1$  and the system (1) has unstable equilibrium with multipliers  $\{\mu_1, \ldots, \mu_m\} \subset \lfloor -\mu^*, 1 \rfloor$ . Let the *N* value be chosen from the condition  $\cot^2 \frac{\pi}{4N} > \mu^*$ , and the coefficients  $\theta_1, \ldots, \theta_N$  from the conditions

$$\mu r(\mu) = \mu \sum_{j=1}^{N} \theta_j \mu^{j-1} = T_N \left( \mu \left( 1 - \cos \frac{\pi}{2N} \right) + \cos \frac{\pi}{2N} \right),$$

where  $T_N(x)$  is the first kind Chebyshev polynomial of order N. Then this equilibrium will be locally asymptotically stable equilibrium of the system (2) (modulo a finite number of cases).

*Proof.* Note that  $r(0) = T_N(\cos \pi/2N) = 0$ ,  $r(1) = T_N(1) = 1$ . In addition  $|T_N(x)| \le 1$ , at  $|x| \le 1$ , whence  $|\mu r(\mu)| \le 1$  at  $\left| \mu \left( 1 - \cos \frac{\pi}{2N} \right) + \cos \frac{\pi}{2N} \right| \le 1$ . The last inequality is equivalent to  $-\cot^2 \frac{\pi}{4N} \le \mu \le 1$ , which proves the theorem.

Note that  $\mu^* \to \infty$  ( $N \to \infty$ ) with asymptotics  $\frac{16}{\pi^2} N^2$ .

4.6. **The general case.** Using the ideas from Theorem 4.1 cases, we can propose the following *T*-cycle stabilization scheme, for which the coefficients  $\theta_j$  are not necessarily constants:

a) find the matrix f'(x), b) find the vectors  $f^{(s)}(x)$ , s = 1, ..., T - 1, c) find the matrix  $f'\left(f^{(T-1)}(x)\right) \cdot ... \cdot f'(f(x)) \cdot f'(x)$ d) find the matrix characteristic polynomial  $\sum_{j=1}^{m+1} \theta_j(x) \mu^{j-1}$ , e) normalize the characteristic polynomial  $\frac{1}{\sum_{j=1}^{m+1} \theta_j(x)} \sum_{j=1}^{m+1} \theta_j(x) \mu^{j-1}$ ,

*f*) build the control system

$$x_{n+1}=F\left( x_{n}\right) ,$$

where

$$F(x) = \frac{1}{\sum_{j=1}^{m+1} \theta_j(x)} \sum_{j=1}^{m+1} \theta_j(x) f^{((j-1)T+1)}(x)$$

or

$$F(x) = f\left(\frac{1}{\sum\limits_{j=1}^{m+1}\theta_j(x)} \left(\theta_1 x + \sum\limits_{j=2}^{m+1}\theta_j(x)f^{((j-1)T)}(x)\right)\right)$$

Let us consider how this scheme looks like in the case of a linear problem. Let f(x) = Ax where A is a non-degenerate  $m \times m$  matrix. Then  $\eta = 0$  is a single fixed point, in the absence of any higher order cycles. We choose  $\theta_j$  from the condition  $\frac{1}{\det(I-A)}\det(\mu I - \mu I)$ 

$$A) = \sum_{j=1}^{m+1} \theta_j \mu^{j-1}.$$
 Then the control system is  $x_{n+1} = \frac{1}{\sum_{j=1}^{m+1} \theta_j} \sum_{j=1}^{m+1} \theta_j A^j x_n.$  By the Hamilton-

Cayley theorem it follows that this system right-hand side is an identical zero.

In the general case, applying this method to stabilizing chaotic motion tending to mixing, one can expect that after a certain number of iterations the trajectory falls into the basin of attraction for the stabilized cycle. Then the convergence to the cycle will be superlinear.

Note that if in all the considered cases  $|\theta_j|$  is being used instead of  $\theta_j$ , then it becomes possible to stabilize the system (1) cycles with multipliers lying in  $M = \mathbb{D} \cup \{\mu : \text{Re}(\mu) \le 0\}$ . Moreover, the convex invariant set of system (1) will remain such for system (2). In addition, the system (2) multipliers, corresponding to those multipliers of system (1) that lie in the unit circle, will become closer to zero.

#### 5. EXAMPLES

Let us illustrate the effectiveness of the generalized predictive control method for finding periodic orbits with several well-known examples of scalar and vector chaotic systems [23]. We have experimented with various number of cycles (28, 50, 101, 1001, etc.) using Maple and Python. The results we include here are all for T = 101 (sections 5.1– 5.12), with a precision of 250 decimals. More results, including the Python code to replicate all results, can be found in the Appendix. It is essential to note that stabilization took only a few iterations (less than 10) for the majority of the systems.

The scheme applied for the logistic and triangular mappings (examples from sections 5.1–5.2) was a general scheme

$$\begin{cases} x_{n+1} = \frac{\theta(x_n)}{1+\theta(x_n)} f(x_n) + \frac{1}{1+\theta(x_n)} f^{(T+1)}(x_n), \\ \theta(x_n) = -f' \left( f^{(T-1)}(x_n) \right) \cdot \ldots \cdot f'(f(x_n)) \cdot f'(x_n). \end{cases}$$

It was possible to find a large number of cycles for all considered periods *T*; in general, different initial conditions are producing different cycles. Numerical calculations show that with sufficiently dense initial values grid, all cycles of a given length can be found. However, in this case it is necessary to ensure that the point  $x_n$  remains within the invariant set, otherwise, as a rule it goes to infinity. If we use  $|\theta(x)|$  instead of  $\theta(x)$ , the point  $x_n$  will always remain in the invariant set. However, in this case we can find cycles only with multipliers from the set  $M = \mathbb{D} \cup {\mu : \text{Re}(\mu) \le 0}$ .

In the two-dimensional case, the scheme used was

(6) 
$$x_{n+1} = f\left(\frac{\theta}{1+\theta}x_n + \frac{1}{1+\theta}f^{(T)}(x_n)\right).$$

The value  $\theta$  should be chosen according to the condition

$$x\left(\frac{\theta}{1+\theta}+\frac{1}{1+\theta}x\right)^T\in\mathbb{D}$$

at  $x = \mu_j^*$ , where  $\mu_j^*$  are cycle multipliers (j = 1, 2), and in general, they are unknown. In the examples below, one of the multipliers never exceeds one in magnitude, while the second one is negative, greater than one in absolute value.

The Theorem 4.2 guarantees stability conditions even if the parameter  $\theta$  satisfies

$$x\left(\frac{\theta}{1+\theta}+\frac{1}{1+\theta}x\right)^T \neq 0.$$

It is enough have  $\theta$  in the neighborhood of multiplier. In general Theorem 4.2 does not provide the estimate on the parameters, however when one multiplier is in the unit disc and the other is real and negative an elementary trial works quite effectively.

Thus  $\theta > 0$ , and we only have to check the compliance with the condition for the second multiplier

(7) 
$$\left| \mu_2^* \left( \frac{\theta}{1+\theta} + \frac{1}{1+\theta} \mu_2^* \right)^T \right| < 1.$$

Let  $\theta = |\mu_2^*| + \Delta$  and assume that  $|\mu_2^*| < 2^T$ . If required that  $\left|\frac{\theta}{1+\theta} + \frac{1}{1+\theta}\mu_2^*\right| < \frac{1}{2}$ , which is equivalent to  $-\frac{1}{3}(1+|\mu_2^*|) < \Delta < 1+|\mu_2^*|$ , or  $\theta \in \left(\frac{2}{3}|\mu_2^*| - \frac{1}{3}, 2|\mu_2^*| + 1\right)$ . Now, if  $\theta < \frac{2}{3}|\mu_2^*| - \frac{1}{3}$  then  $2\theta \le 2|\mu_2^*| + 1$ . Therefore, choosing  $\theta = 2^k$  subsequently for k =1, 2, ... we are sure that for some k we get  $\theta \in \left(\frac{2}{3}|\mu_2^*| - \frac{1}{3}, 2|\mu_2^*| + 1\right)$ , then the condition (7) will be satisfied. Thus, the grid for sorting parameter  $\theta$  should be chosen rather coarse. This justifies the procedure we used in our examples: running (6) with small values for  $\theta$ and then doubling them until obtaining required cycles. To our surprise, the procedure turns out to be quite efficient. However, this idea has been successfully used recently in adaptive interior-point methods by Lesaja and Potra [15].

Therefore, the scheme (6) allows finding cycles both with small multipliers (examples from sections 5.3–5.8) and with large ones (examples from sections 5.9–5.12). In general, the large value of multipliers is not the main obstacle. More challenging is the problem of small basins of attraction for long cycles. Therefore, it is convenient to either select a dense grid for initial values or use a sufficiently large number of iterations so that the point  $x_n$  would fall into the desired basin of attraction. One can achieve any acceptable accuracy in determining the cyclic point. Our subsequent experiments used precision 250 decimals.

5.1. Logistic mapping. The logistic mapping

(8) 
$$x_{n+1} = hx_n (1 - x_n)$$

is, perhaps, the most popular example. Let us consider the case h = 3.99, T = 101. Figure 1 illustrates one of its numerous T = 101-cycles.



FIGURE 1. A 101-cycle of the logistic mapping (8).

5.2. **Triangular mapping.** Our next example is the triangular mapping:

(9) 
$$x_{n+1} = h(1 - |2x_n - 1|), \quad h = 0.99.$$

Figure 2 shows a T = 101-cyclic point.



FIGURE 2. A 101-cycle of triangular system (9).

## 5.3. **Burgers mapping.** For the Burgers mapping:

(10) 
$$x_{n+1} = ax_n - y_n^2, y_{n+1} = by_n + x_n y_n, a = 0.75, b = 1.75$$

a 101-cyclic point is illustrated in Figure 3.



FIGURE 3. A 101-cycle of the Burgers system (10).

5.4. **Tinkerbell mapping.** The Tinkerbell mapping: (11)  $x_{n+1} = x_n^2 - y_n^2 + ax_n + by_n$ ,  $y_{n+1} = 2x_ny_n + cx_n + dy_n$ , a = 0.9, b = -0.6, c = 2.0, d = 0.5

has a 101-cyclic point illustrated in Figure 4.



FIGURE 4. A 101-cycle of the Tinkerbell mapping (11).

## 5.5. **Gingerbredman mapping.** The Gingerbredman mapping:

(12) 
$$x_{n+1} = 1 + |x_n| - y_n, \ y_{n+1} = x_n$$

has a 101-cyclic point represented in Figure 5.



FIGURE 5. A 101-cycle of the Gingerbredman mapping (12).

5.6. **Prey-predator mapping.** For the prey-predator mapping: (13)  $x_{n+1} = x_n \exp(a(1-x_n) - by_n), y_{n+1} = x_n(1 - \exp(-cy_n)), a = 3, b = 5, c = 5$ 

a corresponding 101-cyclic point is illustrated in Figure 6.



FIGURE 6. A 101-cycle of the prey-predator system (13).

5.7. **Delayed logistic mapping.** Figure 7 shows a 101-cyclic point of the delayed logistic mapping:

(14) 
$$x_{n+1} = hx_n (1 - y_n), y_{n+1} = x_n, h = 2.27$$



FIGURE 7. A 101-cycle of the delayed logistic mapping system (14).

## 5.8. Hénon mapping. The Hénon mapping:

(15) 
$$x_{n+1} = 1 + ax_n^2 + y_n, y_{n+1} = bx_n, a = -1.40000001, b = 0.30000002$$

has a 101-cyclic point represented in Figure 8.



FIGURE 8. A 101-cycle of Hénon system (15).

# 5.9. Elhadj-Sprott mapping. The Elhadj-Sprott mapping:

(16) 
$$x_{n+1} = 1 + a \sin x_n + b y_n, \ y_{n+1} = x_n, \ a = -4.0, \ b = 0.9$$

has a 101-cyclic point illustrated in Figure 9.



FIGURE 9. A 101-cycle of the Elhadge-Sprott system (16).

5.10. Lozi mapping. The Lozi mapping:

(17) 
$$x_{n+1} = 1 + a |x_n| + by_n, y_{n+1} = x_n, a = -1.7, b = 0.5$$

has a 101-cyclic point shown in Figure 10.



FIGURE 10. A 101-cycle of the Lozi system (17).

5.11. **Ikeda mapping.** The Ikeda mapping is given by the equations:

(18) 
$$x_{n+1} = 1 + u (x_n \cos \tau_n - y_n \sin \tau_n), \ y_{n+1} = u (x_n \sin \tau_n + y_n \cos \tau_n),$$

where u = 0.9,  $\tau_n = 0.4 - \frac{6}{1 + x_n^2 + y_n^2}$ . The mapping has a 101-cyclic point illustrated in Figure 11.



FIGURE 11. A 101-cycle of the Ikeda system (18).

## 5.12. Holmes cubic mapping. The Holmes cubic mapping:

(19) 
$$x_{n+1} = y_n, y_{n+1} = ax_n + by_n - y_n^3, a = -0.2, b = 2.77$$

has a 101-cyclic point shown in Figure 12.



FIGURE 12. A 101-cycle of the Holmes cubic system (19).

#### 6. CONCLUSION

This article deals with the problem of stabilization for nonlinear systems of two categories: those unstable and those with a priori unknown periodic orbits at discrete time. A well-known method of stabilizing controls, called the predictive control method, first proposed by B.T.Polyak, have been thoroughly investigated in this work. We have found that this method has several disadvantages: it is necessary to know the cycle exact multiplier or its sufficiently accurate estimate even in the scalar case; in the vector case, one must know the whole cycle Jacobi matrix; consequently, the proposed control does not have the required robustness with respect to the system parameters perturbations; the control gain coefficients have different signs, which can trigger the initial system multiplier's shifting beyond the central unit circle (where it lies) when applying the control; therefore, the gain coefficients must be small, and, in order to evaluate them at every instance, we need to know the multiplier's value.

All these shortcomings imply the necessity to modify the predictive control method. We propose not only to use the first and last iterations of the original mapping, but also all previous ones, by considering their linear combination. This linear combination's coefficients are sought as being of a special polynomial, characterized by certain properties. As a result, it was possible for us to extend the predictive control scope. In addition, if the coefficients are non-negative, then for the initial system cycle multipliers lying in the central unit circle the corresponding multipliers of the control system cycle become closer to zero. An algorithm is given as a special case of this method application, for finding a cycle of a given length when its multipliers are known. One of the possible directions for future research is related to investigating new control schemes that combine the use of control system previous states and the initial system predicted states, i.e. the predictive control shall be considered together with the semi-linear control [8] as follows:

(20)  
$$\begin{cases} X_n = \sum_{j=1}^{N_1} a_j x_{n-jT+T} \\ Y_n = \sum_{j=1}^{N_2} b_j x_{n-jT+1} \\ F(x) = \sum_{j=1}^{N_3} \theta_j f^{((j-1)T+1)}(x) \\ x_{n+1} = (1-\gamma)F(X_n) + \gamma Y_n \end{cases}$$

where  $\sum_{j=1}^{N_1} a_j = 1$ ,  $\sum_{j=1}^{N_2} b_j = 1$ ,  $\sum_{j=1}^{N_3} \theta_j = 1$ . Clearly, the *T*-cycles of systems (1) and (20) coincide. The conditions of the system (20) *T*-cycle local asymptotic stability can be for-

coincide. The conditions of the system (20) *I*-cycle local asymptotic stability can be formulated as  $\begin{bmatrix} ( & ) \end{bmatrix}^T = (\overline{a}) + I (\overline{a}) \end{bmatrix}^*$ 

$$\mu_{j} [r(\mu_{j})]^{T} \in (\mathbb{C} \setminus \Phi(\mathbb{D}))^{T}, \quad j = 1, \dots, m,$$
  
$$\Phi(z) = (1 - \gamma)^{T} \frac{z(q(z))^{T}}{(1 - \gamma p(z))^{T}}, \quad q(z) = \sum_{j=1}^{N_{1}} a_{j} z^{j-1}, \quad p(z) = \sum_{j=1}^{N_{2}} b_{j} z^{j-1},$$

where  $\overline{\mathbb{C}}$  is an extended complex plane, and the asterisk denotes the reciprocal operation:  $(z)^* = \frac{1}{\overline{z}}$ .

The semilinear control method (when  $N_3 = 1$  in (20)) has also certain disadvantages [8,9]. Further studies shall aim to eliminating (reducing) inherent disadvantages of predictive control and semi-linear control, synthesizing these approaches together.

### REFERENCES

- [1] Andrievsky B. R., Fradkov A. L. Control of Chaos: Methods and Applications. I. Methods, Avtomat. i Telemekh., (2003), no. 5, 3-45.
- [2] Bartuccelli M., Constantin P., Doering C.R., Gibbon J.D., Gisselfalt M., Hard turbulence in a finite dimensional dynamical system, Phys. Lett. A 142 (6–7) (1987) 349–356.
- [3] Chen G., Dong X. From chaos to order: Methodologies, Perspectives and Application. World Scientific, Singapore (1999)
- [4] Davidchack R.L., Lai Y.-C., Klebanoff A., Bollt E.M., Towards complete detection of unstable periodic orbits in chaotic systems, Phys. Lett. A 287 (2001), 99–104.
- [5] Devaney R L. An Introduction to Chaotic Dynamical Systems. New York: Addison-Wesley Publ. Co., Second Edition., (1993)
- [6] Dmitrishin D., Hagelstein P., Khamitova A., and Stokolos A. Limitations of Robust Stability of a Linear Delayed Feedback Control, SIAM J. Control Optim. 56, (2018), 148-157.
- [7] Dmitrishin D. and Khamitova A. Methods of harmonic analysis in nonlinear dynamics, Comptes Rendus Mathematique, Volume 351, Issues 9-10, 367 - 370 (2013)
- [8] Dmitrishin D., Khamitova A. and Stokolos A. Fejér polynomials and chaos, Springer Proceedings in Mathematics and Statistics, 108 (2014), 49–75.
- [9] Dmitrishin D., Skrinnik I., Lesaja G., Stokolos A. A new method for finding cycles by semilinear control, Physics Letters A, 383 (2019), 1871 – 1878.

- [10] Galias Z. Rigorous investigations of Ikeda map by means of interval arithmetic. Nonlinearity, (2002) 15:1759-1779.
- [11] Jackson E.A. Perspectives of Nonlinear Dinamics. Vol. I, II, Cambridge Univ. Press, Cambridge, 1980, 1990 Chaos II, ed. Hao Bai-Lin. - World Sci., (1990)
- [12] Kittel A, Parisi J, Pyragas K. Delayed feedback-control of chaos by self-adapted delay-time. Phys Lett A 1995;198:433–6.
- [13] Kuntsevich A.V., Kuntsevich V.M. Estimates of Stable Limit Cycles of Nonlinear Discrete Systems, Journal of Automation and Information Sciences, Volume 44, Issue 9, (2012), 1-10
- [14] Kuntsevich A.V., Kuntsevich V.M. Invariant Sets for Families of Linear and Nonlinear Discrete Systems with Bounded Disturbances, Automation and Remote Control, Vol. 73, No. 1, (2012), 83-96
- [15] Lesaja G., Potra F. Adaptive full newton-step infeasible interior-point method for sufficient horizontal LCP, Optimization Methods and Software, 21(1), (2018), doi: 10.1080/10556788.2018.1546857
- [16] Miller J.R., Yorke J.A. Finding all periodic orbits of maps using Newton methods: sizes of basins. Physica D (2000);135:195-211.
- [17] Morgul O. Further stability results for a generalization of delayed feedback control, Nonlinear Dynamics, 1-8 (2012)
- [18] Morgul O. On the stability of delayed feedback controllers, Phys. Lett. A 314 (2003) 278–285.
- [19] Ott E., Grebodgi C., Yorke J.A. Controlling chaos. Phys. Rev. Lett. 64, 1196-1199 (1990)
- [20] Polyak B. T. Stabilizing chaos with predictive control, Automation and Remote Control.66 (11), (2005), 1791-1804
- [21] Polyak B.T., Gryazina E.N., Iterations of perturbed tent maps with applications to chaos control, IFAC Proceedings, Volume 39, Issue 8, 2006, pp 125-129.
- [22] Pyragas K. Continuous control of chaos by self controlling feedback. Phys. Rev. Lett. A 170, 421-428 (1992)
- [23] Sprott, J. C. Chaos and Time-Series Analysis, Oxford University Press, Oxford, Vol. 69 (2003)
- [24] Shalby L., Predictive feedback control method for stabilization of continuous time systems. Advances in Systems Science and Applications. 17 (2017), pp 1-13.
- [25] Ushio T. Limitation of delayed feedback control in nonlinear discrete-time systems. IEEE Trans Circuits Syst I 1996;43:815–6.
- [26] Ushio T., Yamamoto, S.: Prediction-based control of chaos. Phys. Letts. A. 264(1):439-446(1999)
- [27] Vieira de S.M., Lichtenberg A.J. Controlling chaos using nonlinear feedback with delay. Phys. Rev. E 54, 1200-1207 (1996)
- [28] Ypma T.J.. Historical Development of the Newton-Raphson Method. SIAM Rev., (1995) 37:531-551.
- [29] Zhu J, Tian YP. Necessary and sufficient conditions for stabilizability of discrete-time systems via delayed feedback control. Phys Lett A 2005;343:95–107.

### APPENDIX

```
6.1. The Experiments Python Code.
```

```
[3]: # -*- coding: utf-8 -*-
   .....
   Qauthor: Emil
   .....
   # Generalized Predictive Control
   import numpy as np
   import datetime
   import matplotlib.pyplot as plt
   from decimal import Decimal, getcontext, setcontext, ExtendedContext
   #precision
   setcontext(ExtendedContext)
   getcontext().prec = 250
   EPSILON = Decimal(1e-250)
   def cos(x):
       """Return the cosine of x as measured in radians.
       >>> print cos(Decimal('0.5'))
       0.8775825618903727161162815826
       >>> print cos(0.5)
       0.87758256189
       >>> print cos(0.5+0j)
       (0.87758256189+0j)
       .....
      getcontext().prec += 2
       i, lasts, s, fact, num, sign = 0, 0, 1, 1, 1, 1
       while s != lasts:
          lasts = s
          i += 2
          fact *= i * (i-1)
          num *= x * x
          sign *= -1
          s += num / fact * sign
       getcontext().prec -= 2
       return +s
```

```
def sin(x):
    """Return the sine of x as measured in radians.
    >>> print sin(Decimal('0.5'))
    0.4794255386042030002732879352
   >>> print sin(0.5)
    0.479425538604
    >>> print sin(0.5+0j)
    (0.479425538604+0j)
    .....
    getcontext().prec += 2
    i, lasts, s, fact, num, sign = 1, 0, x, 1, x, 1
    while s != lasts:
       lasts = s
       i += 2
       fact *= i * (i-1)
       num *= x * x
       sign *= -1
        s += num / fact * sign
    getcontext().prec -= 2
   return +s
class Map:
   def __init__(self, name, period, theta, niv, initmode, xaxis = 'x',
\rightarrow yaxis = 'y'):
        self.name = name
        self.T = period
        self.t1 = theta
        self.NIV = niv
        self.xaxis = xaxis
        self.yaxis = yaxis
        #create x and y arrays for results (size T+1 x NIV)
        self.x = [[Decimal(0.0)] * self.NIV for j in range(self.T+1)]
        self.y = [[Decimal(0.0)] * self.NIV for j in range(self.T+1)]
        self.foundcycle = [False] * self.NIV
        self.init(initmode)
        self.save_ini()
    def mapname(self):
```

```
return self.name
   def init(self, initmode):#override
       return
   def save_ini(self):
       self.inix = [Decimal(0.0)] * self.NIV
       self.iniy = [Decimal(0.0)] * self.NIV
       for j in range(self.NIV):
           self.inix[j] = self.x[0][j]
           self.iniy[j] = self.y[0][j]
   def fx(self,x,y):#override
       pass
   def fy(self,x,y):#override
       pass
   def g(self, x,y):
       x1 = x
       y1 = y
       for i in range(self.T):
           x2 = self.fx(x1,y1)
           y2 = self.fy(x1,y1)
           x1 = x2
           y1 = y2
       return (x1,y1)
   def run(self, iterations, stopwhenfound = True):
       #perform T+1 initial iterations
       for i in range(self.T):
           for j in range(self.NIV):
               self.x[i+1][j] = self.fx(self.x[i][j], self.y[i][j])
               self.y[i+1][j] = self.fy(self.x[i][j], self.y[i][j])
       #perform it circularly a few more times, recursively (just in ____
\leftrightarrow case)
       for repeat in range(3):
           #init first positions
           for j in range(self.NIV):
               self.x[0][j] = self.fx(self.x[self.T][j], self.y[self.
→T][j])
               self.y[0][j] = self.fy(self.x[self.T][j], self.y[self.
→T][j])
           #perform T+1 initial iterations
           for i in range(self.T):
               for j in range(self.NIV):
```

```
self.x[i+1][j] = self.fx(self.x[i][j], self.y[i][j])
                   self.y[i+1][j] = self.fy(self.x[i][j], self.y[i][j])
       #use averaging, repeat PERIOD_ITERATIONS multiple of periods
       found = False
       for repeat in range(iterations):
           #init first positions
           for j in range(self.NIV):
               (xg,yg) = self.g(self.x[self.T][j], self.y[self.T][j])
               self.x[0][j] = self.fx((self.t1/(self.t1+1)) * self.
→x[self.T][j] + (1/(self.t1+1)) * xg, (self.t1/(self.t1+1)) * self.
→y[self.T][j] + (1/(self.t1+1)) * yg)
               self.y[0][j] = self.fy((self.t1/(self.t1+1)) * self.
→x[self.T][j] + (1/(self.t1+1)) * xg, (self.t1/(self.t1+1)) * self.
→y[self.T][j] + (1/(self.t1+1)) * yg)
           #perform T+1 initial iterations
           for i in range(self.T):
               for j in range(self.NIV):
                   (xg,yg) = self.g(self.x[i][j], self.y[i][j])
                   self.x[i+1][j] = self.fx((self.t1/(self.t1+1)) * self.
→x[i][j] + (1/(self.t1+1)) * xg, (self.t1/(self.t1+1)) * self.y[i][j] +
\leftrightarrow (1/(self.t1+1)) * yg)
                   self.y[i+1][j] = self.fy((self.t1/(self.t1+1)) * self.
→x[i][j] + (1/(self.t1+1)) * xg, (self.t1/(self.t1+1)) * self.y[i][j] +
\rightarrow (1/(self.t1+1)) * yg)
                   if (not self.foundcycle[j]):
                       dx = np.abs(self.x[i+1][j] - self.x[i+1-self.
→T][j])
                       dy = np.abs(self.y[i+1][j] - self.y[i+1-self.
→T][j])
                       if (dx < EPSILON and dy < EPSILON):
                           self.foundcycle[j] = True
                           found = True
                           print(self.name)
                           print('Found ', self.T,'-cycle: initial value
→= ', j, ' (', repeat, 'th iteration)', sep='')
                           print('x = ',self.x[i+1][j])
                           print('y = ',self.y[i+1][j])
           if (stopwhenfound and found):
               break
       return found
   def plot(self,plotinisystem = True):
```

```
for j in range(self.NIV):
            bv = self.foundcycle[j]
            if (not bv):
                continue
            plt.ion();
            fig1 = plt.figure()
            ax1 = fig1.add_subplot(111)
            if (plotinisystem):
                minrepeat = 200 // \text{self}.T + 1
                for k in range(self.NIV):
                    x = self.inix[k]
                    y = self.iniv[k]
                    for count in range(minrepeat):
                        for i in range(self.T):
                            ax1.plot(x, y, marker = 'x', color = 'grey')
                            \mathbf{x}\mathbf{x} = \mathbf{x}
                            x = self.fx(x,y)
                            y = self.fy(xx,y)
            for i in range(self.T):
                ax1.plot(self.x[i][j], self.y[i][j], 'b.')
            ax1.set_xlabel(self.xaxis)
            ax1.set_ylabel(self.yaxis)
            #t = self.mapname() + 'Map: T = ' + str(self.T) + '; initial_
\leftrightarrowvalue ' + str(j)
            t = self.mapname() + ' Map, ' + str(self.T) + '-cycle'
            ax1.set_title(t)
## Logistic mapping ##
class Logistic(Map):
    h = Decimal(3.99)
    def fx(self,x,y):
        return (y)
    def fy(self,x,y):
        return (self.h * y * (1 - y))
    def init(self, initmode):#override
        #init first positions
        self.x[0][0] = Decimal(0.5)
        self.y[0][0] = self.fy(self.x[0][0], self.x[0][0])
        self.x[0][1] = Decimal(0.1)
        self.y[0][1] = self.fy(self.x[0][1], self.x[0][1])
```

```
## Triangular mapping ##
class Triangular(Map):
    h = Decimal(0.99)
    def fx(self,x,y):
        return (y)
    def fy(self,x,y):
        return (self.h * (1 - np.abs(2 * y - 1)))
    def init(self, initmode):#override
        #init first positions
        self.x[0][0] = Decimal(0.5)
        self.y[0][0] = self.fy(self.x[0][0], self.x[0][0])
        self.x[0][1] = Decimal(0.1)
        self.y[0][1] = self.fy(self.x[0][1], self.x[0][1])
## Burgers mapping ##
class Burgers(Map):
    def fx(self,x,y):
        a = Decimal(0.75)
        return (a * x - y**2)
    def fy(self,x,y):
        b = Decimal(1.75)
        return (b * y + x * y)
    def init(self, initmode):#override
        #init first positions
        self.x[0][0] = Decimal(-1.7)
        self.y[0][0] = Decimal(0.2)
        self.x[0][1] = Decimal(-0.5)
        self.y[0][1] = Decimal(0.5)
## Tinkerbell mapping ##
class Tinkerbell(Map):
    def fx(self,x,y):
        a = Decimal(0.9)
        b = Decimal(-0.6)
        return (x**2 - y**2 + a * x + b * y)
    def fy(self,x,y):
        c = Decimal(2.0)
        d = Decimal(0.5)
        return (2 * x * y + c * x + d * y)
    def init(self, initmode):#override
        #init first positions
        self.x[0][0] = Decimal(0.0)
        self.y[0][0] = Decimal(-0.3)
```

```
self.x[0][1] = Decimal(-0.5)
        self.y[0][1] = Decimal(-0.5)
## Gingerbredman mapping ##
class Gingerbredman(Map):
    def fx(self,x,y):
        return (1 + np.abs(x) - y)
    def fy(self,x,y):
        return (x)
    def init(self, initmode):#override
        #init first positions
        for j in range(self.NIV):
            self.x[0][j] = Decimal(-2.09 + 1.5*(j+1)/12.0)
            self.y[0][j] = Decimal(-2.09 + 1.5*(j+1)/12.0)
## PreyPredator mapping ##
class PreyPredator(Map):
    def fx(self,x,y):
        a = Decimal(3.0)
        b = Decimal(5.0)
        return (x * np.exp(a * (1 - x) - b * y))
    def fy(self,x,y):
        c = Decimal(5.0)
        return (x * (1 - np.exp(-c * y)))
    def init(self, initmode):#override
        #init first positions
        for j in range(self.NIV):
            self.x[0][j] = Decimal(0.02 + (j+13)/19.0)
            self.y[0][j] = Decimal(0.1 + (j+13)/11.0)
## DelayedLogistic mapping ##
class DelayedLogistic(Map):
    def fx(self,x,y):
        h = Decimal(2.27)
        return (h * x * (1 - y))
    def fy(self,x,y):
        return (x)
    def init(self, initmode):#override
        #init first positions
        self.x[0][0] = Decimal(0.1)
        self.y[0][0] = Decimal(0.1)
        self.x[0][1] = Decimal(0.05)
        self.y[0][1] = Decimal(0.025)
## Henon mapping ##
```

```
class Henon(Map):
    def fx(self,x,y):
        a = Decimal(-1.40000001)
        return (1 + a * x ** 2 + y)
    def fy(self,x,y):
        b = Decimal(0.3000002)
        return (b * x)
    def init(self, initmode):#override
        #init first positions
        for j in range(self.NIV):
            self.x[0][j] = Decimal(-1.0 + j/9.0)
            self.y[0][j] = Decimal(0.3) * self.x[0][j]
## ElhadjSprott mapping ##
class ElhadjSprott(Map):
    def fx(self,x,y):
        a = Decimal(-4.0)
        b = Decimal(0.9)
        return (1 + a * sin(x) + b * y)
    def fy(self,x,y):
        return (x)
    def init(self, initmode):#override
        #init first positions
        self.x[0][0] = Decimal(10.0)
        self.y[0][0] = Decimal(10.0)
        self.x[0][1] = Decimal(0.0)
        self.y[0][1] = Decimal(0.0)
## Ikeda mapping ##
class Ikeda(Map):
    def fx(self,x,y):
        k = Decimal(0.9)
        a = Decimal(0.4)
        b = Decimal(6.0)
        ff = a - b / (1 + x + 2 + y + 2)
        return (1 + k * (x * cos(ff) - y * sin(ff)))
    def fy(self,x,y):
        k = Decimal(0.9)
        a = Decimal(0.4)
        b = Decimal(6.0)
        ff = a - b / (1 + x + y + y)
        return (k * (x * sin(ff) + y * cos(ff)))
    def init(self, initmode):#override
        #init first positions
```

```
self.x[0][0] = Decimal(0.1)
        self.y[0][0] = Decimal(-1.0)
        self.x[0][1] = Decimal(1.0)
        self.y[0][1] = Decimal(0.0)
## Lozi mapping ##
class Lozi(Map):
    def fx(self,x,y):
        a = Decimal(-1.7)
        b = Decimal(0.5)
        return (1 + a * np.abs(x) + b * y)
    def fy(self,x,y):
        return (x)
    def init(self, initmode):#override
        #init first positions
        #self.x[0][0] = Decimal(-0.5)
        #self.y[0][0] = Decimal(-0.5)
        #self.x[0][1] = Decimal(0.5)
        #self.y[0][1] = Decimal(0.5)
        #for j in range(self.NIV):
           self.x[0][j] = Decimal(0.5 * (-1)**j + j/2.0)
        #
             self.y[0][j] = Decimal(-0.5 * (-1)**j + j/2.0)
        #
        self.x[0][0] = Decimal(0.5)
        self.y[0][0] = Decimal(0.0)
        self.x[0][1] = Decimal(-0.5)
        self.y[0][1] = Decimal(-0.5)
## Holmes mapping ##
class Holmes(Map):
    def fx(self,x,y):
        return (y)
    def fy(self,x,y):
        a = Decimal(-0.2)
        b = Decimal(2.77)
        return (a * x + b * y - y * * 3)
    def init(self, initmode):#override
        #init first positions
        self.x[0][0] = Decimal(0.1)
        self.y[0][0] = Decimal(0.1)
        self.x[0][1] = Decimal(-0.1)
        self.y[0][1] = Decimal(0.1)
## Multihorseshoe mapping ##
class Multihorseshoe(Map):
```

```
def fx(self,x,y):
        ak = Decimal(3.0)
        return (x * np.exp(ak - Decimal(0.8) * x - Decimal(0.2) * y))
    def fy(self,x,y):
        bk = Decimal(3.0)
        return (y * (\text{Decimal}(0.2) * x + \text{Decimal}(0.8) * y) * \text{np.exp}(bk - 1)
 \rightarrowDecimal(0.2) * x - Decimal(0.8) * y))
    def init(self, initmode):#override
        #init first positions
        self.x[0][0] = Decimal(3.0)
        self.y[0][0] = Decimal(6.0)
success = False
#choose mapping(s) from above
allmaps = [
#
     Param: (name, period, theta, niv, initmode)
    Logistic("Logistic", 101, Decimal(2e24), 2, 0, 'x[n]', 'x[n+1]'),
\leftrightarrow#tested: precision = 250
    Triangular("Triangular", 101, Decimal(64e28), 2, 0, 'x[n]',
 \leftrightarrow'x[n+1]'), #tested: precision = 250
    Burgers("Burgers", 28, Decimal(680), 2, 0), #tested: precision = 250
    Burgers("Burgers", 50, Decimal(10000), 2, 0), #tested: precision = 250
    Burgers("Burgers", 101, Decimal(16e7), 2, 0), #tested: precision = 250
    Tinkerbell("Tinkerbell", 28, Decimal(100), 2, 0), #tested: precision
 \rightarrow = 250
    Tinkerbell("Tinkerbell", 50, Decimal(2000), 2, 0), #tested: precision
 →= 250
    Tinkerbell("Tinkerbell", 101, Decimal(8e8), 2, 0), #tested: precision
 →= 250
    Gingerbredman("Gingerbredman", 28, Decimal(30), 2, 0), #tested:
 \rightarrow precision = 250
    Gingerbredman("Gingerbredman", 50, Decimal(100), 2, 0), #tested:
 \rightarrow precision = 250
    Gingerbredman("Gingerbredman", 101, Decimal(4e5), 2, 0), #tested:
 \hookrightarrow precision = 250
    PreyPredator("Prey-predator", 28, Decimal(350), 15, 0), #tested:
 →precision = 250, 8th ini value
    PreyPredator("Prey-predator", 50, Decimal(19500), 15, 0), #tested:
↔precision = 250, 13th ini value, iteration 112
    PreyPredator("Prey-predator", 101, Decimal(1e8), 2, 0), #tested:
 →precision = 250 (206th iteration)
```

```
DelayedLogistic("Delayed Logistic", 101, Decimal(1e5), 2, 0), #tested:
 \rightarrow precision = 250
    Henon("Henon", 28, Decimal(15000), 11, 0),
    Henon("Henon", 50, Decimal(100000), 11, 0),
    Henon("Henon", 101, Decimal(1e16), 11, 0), #tested: precision = 250
    Henon("Henon", 1001, Decimal(5e174), 11, 0), #tested: precision = 250
    ElhadjSprott("Elhadj-Sprott", 101, Decimal(1e28), 2, 0), #tested:
\rightarrow precision = 250
    ElhadjSprott("Elhadj-Sprott", 1001, Decimal(1.5e317), 2, 0), #not
#
\leftrightarrow tested: precision = 355
    Lozi("Lozi", 28, Decimal(4000), 2, 0), #tested: precision = 250
    Lozi("Lozi", 50, Decimal(9.9e7), 2, 0), #tested: precision = 250
    Lozi("Lozi", 101, Decimal(64e16), 2, 0), #tested: precision = 250
    Lozi("Lozi", 601, Decimal(2e120), 2, 0), #tested: precision = 250
    Lozi("Lozi", 1001, Decimal(1e203), 2, 0), #tested: precision = 250
    Ikeda("Ikeda", 28, Decimal(9000), 2, 0), #tested: precision = 250
    Ikeda("Ikeda", 50, Decimal(1.7e7), 2, 0), #tested: precision = 250
    Ikeda("Ikeda", 101, Decimal(1e22), 2, 0), #tested: precision = 250
    Ikeda("Ikeda", 1001, Decimal(3.8e225), 2, 0), #tested: precision = 250
    Holmes("Holmes", 101, Decimal(2e28), 2, 0), #tested: precision = 250
    Multihorseshoe("Multihorseshoe", 1001, Decimal(1.5e187), 1, 0),
\leftrightarrow#tested: precision = 250
٦
success = False
#process all selected
for map in allmaps:
    if (map.mapname == "Elhadj-Sprott" and map.T == 1001):
        getcontext().prec = 355#special case
        EPSILON = Decimal(1e-355)
    #run
    success = map.run(250)
    #plot
    map.plot()
    if (map.mapname == "Elhadj-Sprott" and map.T == 1001):
        getcontext().prec = 250#all others
        EPSILON = Decimal(1e-250)
print('all done')
print(datetime.datetime.now())
```

Logistic

```
Found 101-cycle: initial value = 1 (16th iteration)
```

x = 0.

 $\leftrightarrow 6208441673334530510193712421079700126902660144923404119694819796097082349 \\ 11938072808304801042336167995102899637104914594667846339596919479775167434867788 \\ 53219666198030338615187822832999207491246905623778821254731576093857494224224021 \\ 47797194242586651 \\ \end{cases}$ 

y = 0.

 $\hookrightarrow 9392327820137228002754955380432561475641787058732521657996624880704328735$  15467180570529462820821955349167591152288812597184499607804672705249413697803668 56813350092378812688074975048458091890730194638857008904876155355625587374146225 25168346415796463

Triangular

```
Found 101-cycle: initial value = 1 (34th iteration) r = 0
```

x = 0.

 $\leftrightarrow 2575821524435856633314993107862728612361611569741225558276447680309968781\\66736561270986662776665759766406588690503984057893648676945364568939371331611044\\26296659078376849265035286785034022475383981868424947884934771167584592624272785\\57692563906927813$ 

y = 0.

 $\hookrightarrow 5100126618382996088207904533508069557524375761355335745926115557897996161\\ 83460975127533062975652589982855523697139349840000029513594739145698438795902719\\ 74648985350145959294070589255474085521410750837717614062091114964050749692343686\\ 93672987153253312$ 

Triangular

```
Found 101-cycle: initial value = 0 (34th iteration)
```

x = 0.

 $\leftrightarrow 3491506884527178899645588909615486544805272152132746957003654283560059448\\42118842291992898645167486956507230404107157882186720615311001369810781530647687\\48893632543471185619960471849404214435590805900281979612665697237017739807202735\\84487809135892385$ 

y = 0.

 $\hookrightarrow 6913183631363814159276644699609515631517289581215740207786655006524859851\\19741442264159723262782904340857125235285395747169726024043215833672570544007405\\97602054639102683659289046757076905760997417307051156782730733208325982589440907\\38225106729986306$ 

Burgers

Found 28-cycle: initial value = 1 (153th iteration)

x = -0.

 $\leftrightarrow 189820353603430385414996778555225514814348408752963755155207125274978477$  15333431251975738593113406311985437796728895988541116252960519152805568267712563 54972920544128738217322521049261178731159336344099910811562337947588858739304417 757816610041919377

y = -0.

 $\hookrightarrow 000160108295377578425762294517627034914056637559336515584204233440484163\\ 28071117687003557806801696435117889246075992526577059674099320121303730813631539\\ 41520332460486354035523234989755186957539356654000125036418934366557803210580601\\ 107118787448732162044$ 

Burgers

Found 28-cycle: initial value = 0 (153th iteration)

x = -0.

 $\hookrightarrow 045046925767218743446435475467218503555332898342324281888204233114749501\\62296083072969099053154657624218010672449836251767671058828233865234782251246729\\64907233610810381871972657083998128756241569348215202036647586256404571905150970\\4505871532145621311$ 

y = 0.

 $\hookrightarrow 0018622570870654392363625588935676651829530095228399377525197019997797163\\60282999589850411610001744527104397726632581744900784003486653189103246283615124\\68947748911618394387899148405202316627434033260581296249284236655589110968760832\\1432693776165750676$ 

Burgers

Found 50-cycle: initial value = 0 (57th iteration)

x = -0.

 $\leftrightarrow 017817577260542359802315327228488443365556590968293468661242421348877840\\50005676410670163193895940462991170012439546205856631936126782802178169581551609\\06952948930100845519242962095017879742418948716533080498143539486406102273614427\\9377826562830292326$ 

y = -0.

 $\leftrightarrow 005708247435297464406137161753997573275220045959859992343092833723078136$ 39972652027049371821227814523650373328923032043151564155985745585999102963741756 85801483232212841755103853345694770561635600980254416878900267361168267043454603 30385432006023952171

Burgers

Found 101-cycle: initial value = 0 (20th iteration) x = -0.

 $\hookrightarrow 991286211441260650746717244293092041768562287757457875973371354386316592$ 95866818056958481067770350544245850755740008343373668572239744628660758245902372 46612598138867397562679172858507091629067261746609150656869257598282536923022815 389117352292511784

y = -0.

 $\hookrightarrow 000179865945389048499356746436141587376608956128073922933204009694736066$ 61362022579828900519159308129070119988533380158408089600233653981565148246225015 33554677137277750632099631787844034749562461545577756211308504457544695006943312 436659294495803100011

Tinkerbell

Found 28-cycle: initial value = 1 (37th iteration)

x = 0.

 $\leftrightarrow 1652699415532249905525004948040007221237083079395887690837959586816518844\\ 84178724666819675366586777459776611879395085884379404117097949201133124754116630\\ 50672895457427167599490465225303468676389482234562828860417261132941537281743352\\ 82751848397367355\\ \label{eq:starses}$ 

y = 0.

 $\hookrightarrow 1643681389595992098746955777897269813494538852567047980679089895866921500\\ 84536904125281332635524287237255555391712116584927667827327756330611034346551563\\ 76542590042817275046986359645530774083737316318877994904944403989072937791770847\\ 66564755271282848$ 

Tinkerbell

Found 50-cycle: initial value = 1 (31th iteration) x = 0.

 $\hookrightarrow 1643396243845003512982312095785460058297749540417345545374819517662064751$ 39661757439304419114962721213688271604521582467911398290192539215346548275939964 07783619051949491362210281307472361358710073127504730473226234458804168081154491 10796236481740895

y = 0.

 $\leftrightarrow 1253676436843538676204139706175187961211810443722066059679288488706666204\\38119994090759985394326006878254899470797162154286480035266453993591892945006448\\82811931752656028606915352568030358416352486299986168436281266968133147180985470\\75837187477225340$ 

Tinkerbell

Found 101-cycle: initial value = 0 (11th iteration)

x = -0.

 $\leftrightarrow 003512998306135684573245899575553101958214113166274285463019653362821012 \\ 00310431781731979055622687764644046708338630352399383512496092021969581797054053 \\ 17660055378859438615810744934675551055343144133991068021196156437416657023635980 \\ 228436600111693776$ 

y = -0.

 $\leftrightarrow 325683068982302430220086824513764205064773228250716242837063500280318610\\ 12985442592047614258829952300870748714696707386780737735948630172886318726530546\\ 55949575967814546864681875893745412725987276725497415431715642760540499830248835\\ 245415392952288302$ 

Tinkerbell

Found 101-cycle: initial value = 1 (11th iteration)

x = -0.

 $\leftrightarrow 114468818982283984622986417327106231817790322496221108184728748020434921\\ 81877373496588743856392786956447481181772913636089530370178653527722814252279105\\ 97095236254211764216388842689102855780298883767823513649681201072859972036940830\\ 853673088346092732\\ \label{eq:starses}$ 

y = -0.

 $02537498132296767387709685404195760920208702053953453624176908409301354851640969\\06558232863776852767005716268487525140333702799481819819465457666389682836425949\\186971201894101448$ 

Gingerbredman

Found 28-cycle: initial value = 0 (126th iteration) x = 1.

y = -0.

Gingerbredman

Found 50-cycle: initial value = 1 (100th iteration)

x = 5.

 $\leftrightarrow 1734104046242774566473988439306358381502890173410404624277456647398843930\\ 63583815028901734104046242774566473988439306358381502890173410404624277456647398\\ 84393063583815028901734104046242774566473988439306358381502890173410404624277456\\ 64739884393063583815028901734104046242774566473988439306358381502890173410404624277456\\ 6473988439306359$ 

y = 3.

 $\hookrightarrow 8092485549132947976878612716763005780346820809248554913294797687861271676$ 30057803468208092485549132947976878612716763005780346820809248554913294797687861 27167630057803468208092485549132947976878612716763005780346820809248554913294797 6878612716763006

Gingerbredman

```
Found 101-cycle: initial value = 0 (47th iteration)
```

x = 1.

 $\hookrightarrow 5201913339864814799568592904835217308974749512197129990202777800647110642$ 74717403653787573170430501469583329902933403587923894519318640244354247795625005145599894618114158221022039633468628306562492281600158072828762668466940549797057540156261577600

y = 2.

 $\leftrightarrow 3412183133958489416530136749462799371001868881881725299062265875204794875$ 80580094349719667717741205140660118719280768629129858475420498423388192289009821 92107884705630521228686925236491771156648526711838172941554218156969612145262343 2650272099322427

Prey-predator

```
Found 28-cycle: initial value = 8 (40th iteration)
```

x = 0.

 $\hookrightarrow 5953755523758989238023479395418373846102282148598660226863639433586297547$ 

y = 0.

 $\leftrightarrow 0211059237258230412444800968949072001501837832485441049486828582258614143$ 42177134129146694817179375072914085201833641590728926071358290731923139180518762 26206862755324004907441229516056836992499650472336985928502183690576800281561777 448294032851423658

Prey-predator

Found 50-cycle: initial value = 13 (109th iteration)

x = 0.

 $\hookrightarrow 7142105897277822691465106056672195672771983736022319507011869671712743210 \\ 61818513424495560913110214953367927639717553246372671682064189361326057579592417 \\ 21816835144125351174438466953527401867118380852979182324948372142957088967229892 \\ 76864050793796989$ 

y = 0.

 $\hookrightarrow 0177136810593335252633126015980846540605489473687784254294450792228692499$  25060994735838044243467821404541369659979550131508032750268753481221884836071262 57266705880610187800636723418180668868536502115129783620704838194055316239928073 667471166960210315

Prey-predator

Found 101-cycle: initial value = 0 (206th iteration)

x = 1.

 $\leftrightarrow 9260259093313495184502822115024639129044551920664860718565403695867826398\\ 87223418260288745024327723658316935691320149500153577582620674080718933225519751\\ 07029759204174711903855679027975374036261434887090622240021346220833167730226530\\ 1895640391270805 \\ \end{tabular}$ 

y = 0.

 $\leftrightarrow 0819656977491978025534665153704321459460247600824057371334659066569499929$  19415382515844192482324280511042618267841590667062343839178532299590974140239671 94888583153397977676296702313914798330586647581179023264426223618440430042625474 921717452656551351

Delayed Logistic

Found 101-cycle: initial value = 0 (16th iteration) x = 0.

 $\hookrightarrow 1678748017303368148664731739230001516170520621102573342694914705791203350\\53577270755130828117125757605049755725900181055839625047853336817362793758180829\\52579966265424136881617874494978285372762605727394331816921099367231742143336568\\64795807078531509$ 

y = 0.

 $\hookrightarrow 0770151302304203122539515367646238602366665192317201462953476921443384306\\53538219945941057871032141589587495130780627635448059811398788765786918586297107\\04511491007198288576286401514236398744733174139065947038678134727354930003981679$ 

Henon

```
Found 28-cycle: initial value = 9 (17th iteration) x = 0.
```

 $\leftrightarrow 1225274379717909569407593957713277479981395912599237257370472882802515127$ 47436818023839554634184358469786556952888130735300916622178202741924882739359890 79654693343833141107877615270270287810400998509935157298686345637766666218125510 42861393886213255

y = 0.

 $\hookrightarrow 2404705748617771006741812644243663912860393192675163045264599241765456761 \\ 03384515850118465034030287266830661843908904709768043143666598101547313952460578 \\ 9856351678900801978701435033339748567763785342899978071927709003251611994018855 \\ 36723147097399276$ 

Henon

Found 50-cycle: initial value = 4 (28th iteration)

x = 0.

 $\leftrightarrow 3996970575491603269611853889807529449382090970683725111031127401007267040\\91852249194344152926298003589922172641014310499870988519799793166918480282404881\\87851128856696125919360321543380942906435153189876581689879134409608959793419252\\85683214295659066$ 

y = 0.

 $\leftrightarrow 1858157650867404178678987172893942811317962843352428624346445724025625505$  17605074642000793084024114216894937466775014973688795528719606656964743957342273 19143800603093187504460969387811699454853978276110087969251328896688587849276814 04730067731535688

Henon

Found 101-cycle: initial value = 3 (5th iteration)

x = -1.

 $\hookrightarrow 060258444222247866742638810996174712237994304086139058405204584406574936\\10054937610707398549833943592413859024041951723784575259599584624319112121775558\\55940576967491549722929584761057613335208353202151315618354461483363894542771233\\33630165258629542$ 

y = 0.

 $\leftrightarrow 3677063137022115095162724391212992569158284967816822516801335792604248299\\ 48163697837362397172747931265759383366953570511425274031170629332405290054885386\\ 44476645017240897744266483387558067298357358481009906146867205897435653992237195\\ 49692505945369248$ 

Henon

Found 101-cycle: initial value = 1 (5th iteration)

x = -1.

 $\hookrightarrow 223394105864755510373417002250700700832245916019699990570706239332857579$  35581915968796030648831218048575290254120428806531147042438139830363159835921516 91213560975389758239217860632792518460628622525689127569936865565764531154384838

y = 0.

 $\leftrightarrow 3735391160921831697195819411098258360079937910920758863707998430802931324$ 96372788046578187596273640196530201700009201840176042749307272301258250663582423923244541801822612505942671535412127394606109272084594562021556919367230390191247846578349192982

Henon

Found 1001-cycle: initial value = 8 (1th iteration) x = -0.

 $\hookrightarrow 592743453771613574925072767996497608118639264643556222066064582174528385$ 33930070711040456467157275317528377284227451504904426397934492766451996075674066 75590869380828870563915585892488676550872160201589808325840358291695848113390832 198314751325737388

y = -0.

 $\hookrightarrow 278599124740739223552490480783323864825130411431783945940598171923110747\\68481062690775860955120055638629720220251760694790281847238344666972576967818309\\59962865296988796372494257566312304274418148998461227453299915867965315197632977\\391483881135875051$ 

Henon

Found 1001-cycle: initial value = 7 (1th iteration)

x = 0.

 $\leftrightarrow 8941919732891004964148343848269216565892784644694368028156989856013466207\\67661763990194416782686744283837501353224502507172453520468420810727603395288781\\23346409446022514275646650956376483675885096786648355097907374892038723451274883\\90659895832318884$ 

y = -0.

 $\hookrightarrow 166587525625092390527118759356461277084252076913759235507669550647902490 \\ 04959728447290195711916578804533567695850984144216364043179282900874295441944802 \\ 74798007921713212081970279826172577369155233406031454311009931313636076648374498 \\ 408454518154853189$ 

Elhadj-Sprott

Found 101-cycle: initial value = 0 (28th iteration) x = 10.

 $\hookrightarrow 638042667340165574037086832523736931294231649777353960140029079003235466$ 26466130742235350280124668054958468730422977571505996194661867034950396222843249 69848479617305046067954828394489250017027213523535684858637421669352817809001954 0233031811697283

y = 9.

 $\leftrightarrow 7976244555120301908568785476949660602394491099418407608095174460853620708\\85065934500464284216209437311394916480615559604236086522380237763388995451290328\\68976966882156120826108555517992730626447110970553913041316708062018449525134378\\1709115461733291$ 

Lozi

Found 28-cycle: initial value = 0 (120th iteration)

x = 0.

 $\hookrightarrow 6715218431136381974485379050843528421662407309957097024023915199465078077$ 76245989403195320206207522317731542254977643461754862064942661529593558184809541 64310321666952963490247821623265262309305412318747764733800278036909952460261337 84997805396061125

y = -0.

 $\leftrightarrow 458193561596474123291808927822656067571143717038126774970696812554112537$ 7113004733966619131890176967579059625555401289846366037294170129010749198654331587263734579141242673312717268080626029071990480309496362331407056598647344450855501253645934866244

Lozi

Found 50-cycle: initial value = 0 (34th iteration) x = 0.

 $\leftrightarrow 7243138612972183930487886387940489931010042662371382123839134515783729788$  27810597373893914575351283081345621642157388105169338607198095013139387558826256 57298139258402453026335440704134831281782899147224146764686651891441214116031302 44524606856164691

y = -0.

 $\leftrightarrow 421592048069018469356537053566885850385300304297589313030242662394345693$  17508281837753291463737258533607344417489794663861305640955484555313939640429453 52721313572346663596027470322178613425213009760843183345818536152899884551805340 426751311017044060

Lozi

Found 101-cycle: initial value = 0 (7th iteration) x = 1.

 $\hookrightarrow 0279935471366691534344603188283232642084959647976241375394632755600110970$ 63086524552422840174830317891266535556496267170961374055458396052622143596180475 48004289855650505457236674241535309303372647124700724818806459217959970612420717 4469454086970554

y = 0.

 $\hookrightarrow 0863943028053452768653172895754238752920110467943741271105322465077075763$ 9941854563522979415904218614886339651918301053236100013168210618526418262512872624918071501464936525684191835968589631708029744850865563081874610195030360671788412427755104146957

Lozi

Found 601-cycle: initial value = 1 (1th iteration)

x = -0.

 $\hookrightarrow 677303452427627896508055914368165636763365583437734260157201148735352859$ 7790209696577910547842315002927041951636402399126346474119867637560462228594551597230290766516148580334276339726091881133085710297730114598985869262207931871254286737940595218300

y = 0.

 $\leftrightarrow 9239803462416223688824712290328801260835728317838954101919478349525183045558086478613477733094740180753605756904737620333090200363762669303585375386467064691419641852158592653349954553706198066440310378570858797918109662767732858646042418137098023711$ 

Lozi

Found 1001-cycle: initial value = 0 (1th iteration) x = -0.

 $\leftrightarrow 131617250269451507405655236417895104325535663402424630969147088080013679$ 36723627314750676503214858853487210486379766212312772583446219445417222555423796 51317969140679415718311175941421024012613186996840882673792234970756212020706489 736075642448637349

y = 0.

 $\hookrightarrow 7477154051233377942551663788887827423493296433942507044747582356100178501$  22942538129095539994368319536784967739397867846794695981689654450495940228852966 11751137849298555223925203114593802340508628757418158866561499399710926911750209 90139122788672961

Ikeda

Found 28-cycle: initial value = 0 (79th iteration)

x = 1.

 $\leftrightarrow 4011854198924391287479391443932094588433219419621375330089116293790595040\\ 84511852856534909149094894143575807902223096814885137873856077265927346067420864\\ 80586880926235477705671358391979219437910557297155995900238457318984110007697315\\ 5647603299381155$ 

y = -0.

 $\leftrightarrow 052160829575268621799448640135737366473920398510332449444924751432651696\\54121986031279223273823891755774210251131088356267586368594620330212625736807666\\80337871277452677616260550657165603733670530418192957819670837509426746954594780\\4875470672503084141$ 

Ikeda

Found 50-cycle: initial value = 0 (196th iteration) x = 1.

 $\hookrightarrow 1785614536086387507088522807427851283801416010486260342756456563157920523\\ 66358338815395442292236789635057190791589182728573452937881204898672163868731093\\ 92062974417789985835052543266771052009578906071594615817631199152496495642966935\\ 8729123889193085$ 

y = -1.

 $\leftrightarrow 199983994024039868495574156992408632143302494249312907225803455945802732\\65151101036790076964584094579358473725875933098233635277199572662073560688231567\\54733877543969372111335263593710998037951694330915342548188101959682890306459976\\32062714808315172$ 

Ikeda

Found 101-cycle: initial value = 0 (11th iteration)

x = 1.

 $\leftrightarrow 2621664717835257387084098668590556481101917958116279497048364940796280035\\64261085283177763399869191744030478701485227646322256878341569201076897345500014\\73128989313971548200346915874784294808101778588695993392905771040789473268785139\\9783082826163767$ 

y = -0.

 $\leftrightarrow 132488738796657124571978486438331496544955067646527762670189035238081194\\ 45797388411889517601779164368600122982278424294010433147031397790570779423842637\\ 79171540776436031191215657354936480428160945189362346451201193788418354293788221\\ 713730646035869555$ 

Ikeda

Found 1001-cycle: initial value = 1 (2th iteration) x = 1.

 $\leftrightarrow 4112894130560773250880430953382273104815592378825068591299094417379132603\\86563929218988549909460544360426114596378153860079818885788806545266347319634687\\28255415975766033921728385239245625628664436926638075023742523965727662642460959\\6651334143685892$ 

y = -0.

 $\leftrightarrow 186881957730856360669522421029139840244013727018630446263480977933798152\\51276944783656875294342331347003307184214529852940311205390573872096014121981105\\19699614738875044012560578388148317663432079279373423138207845269583247640670610\\409089629668680995$ 

Ikeda

Found 1001-cycle: initial value = 0 (2th iteration)

x = 1.

 $\leftrightarrow 1513099706312689328185459311763371777378832181301990045558051927931206159\\ 40907642949932051429447518116421948348122454315783859570071049696677483869416894\\ 99767772601239050311268974830001217523906750055027041492630657324487016074599948\\ 6757284753203324$ 

y = -1.

 $\leftrightarrow 221126110386722130978592080817830894573886645506244755184655690735405788$ 778485116871064153233032224320150522383765839704852014573280970784128056817904176620827495366721696893361156931923845100649875100664574869965693766591770878829125589582500164099

Holmes

Found 101-cycle: initial value = 0 (6th iteration)

x = 1.

 $\hookrightarrow 5637030822347211045173865285834111900443612731692169454286218950310653237\\ 61082962840975004387217939157740706408417934839975330699762921923881851268095804\\ 50391465882437179830214790886273016033689600728518986040610712798492617878975100\\ 2934104819249006$ 

y = 0.

 ${\hookrightarrow} 2810347246457011994936574847299510655538129425364130502653147521295414841$ 

Multihorseshoe

Found 1001-cycle: initial value = 0 (6th iteration) x = 3.

 $\hookrightarrow 0753840675646298388604991543803516600693282471858613210573387862392795762\\53147145066290940867422437246422151371987486692925640745390254473264157951972871\\66338696181524422143531944208080649681258843046826741888229877988354070869141022\\0130079145171214$ 

y = 0.

 $\leftrightarrow 3546597755462393715479899686148861980759521204156354748133847995460261349\\54032251180963251183866364315863145121351987463618461708358704130176132006788589\\34415733244953338245847609521508657884829509861178593465110909308211433033386764\\58736199942181651$ 











Burgers Map, 28-cycle

































































## []:

DEPARTMENT OF APPLIED MATHEMATICS, ODESSA NATIONAL POLYTECHNIC UNIVERSITY, ODESSA 65044, UKRAINE

*E-mail address*: dmitrishin@opu.ua

DEPARTMENT OF MATHEMATICAL SCIENCES, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460, USA

*E-mail address*: ieiacob@GeorgiaSouthern.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, GEORGIA SOUTHERN UNIVERSITY, STATESBORO, GA 30460, USA

*E-mail address*: astokolos@GeorgiaSouthern.edu