SOME EXTREMAL PROBLEMS FOR TRINOMIALS WITH FOLD SYMMETRY

DMITRIY DMITRISHIN, DANIEL GRAY, AND ALEX STOKOLOS

ABSTRACT. The famous T. Suffridge polynomials have many extremal properties: the maximality of coefficients when the leading coefficient is maximal; the zeros of the derivative are located on the unit circle; the maximum radius of stretching the unit disk with the schlicht normalization $F(0) = 0$, $F'(0) = 1$; the maximum size of the unit disk contraction in the direction of the real axis for univalent polynomials with the normalization $F(0) = 0$, $F(1) = 1$. However, under the standard symmetrization method $\sqrt[T]{F(z^T)}$, these polynomials go to functions, which are not polynomials. How can we construct the polynomials with fold symmetry that have properties similar to those of the Suffridge polynomial? What values will the corresponding extremal quantities take in the above-mentioned extremal problems? The paper is devoted to solving these questions for the case of the trinomials $F(z) = z + az^{1+T} + bz^{1+2T}$. Also, there are suggested hypotheses for the general case in the work.

Keywords. Suffridge polynomials, polynomials symmetrization, domain of univalence of trinomials with fold symmetry, extremal univalent trinomials with fold symmetry.

1. INTRODUCTION

This work is motivated by problems of classical geometric complex analysis, which studies various extremal properties of the univalent (or similar in properties to univalent) in the central unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ functions $F(z)$ with different normalizations.

One of the first fundamental works in this theory was the 1916 paper by Ludwig Bieberbach, in which he proved the exact estimate for the

1

second coefficient of the univalent in $\mathbb{D}$ functions of the form

(1)
$$F(z) = z + \sum_{j=2}^{\infty} a_j z^j,$$

namely, $|a_2| \leq 2$. This estimate immediately implies the famous Koebe $1/4$ theorem. In that same work, Bieberbach stated in a footnote that in the general case there must hold $|a_j| \leq j$. This innocently-looking statement became known as the Bieberbach conjecture, which, since then, has been the engine of the development of geometric complex analysis for 105 years. In 1984, the Bieberbach conjecture was proved by Louis de Branges, who also showed that the extremal function, up to rotation, is the Koebe function

$$K(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \dots .$$

Traditionally, the class of univalent in $\mathbb{D}$ functions of the form (1) is denoted by the symbol $\mathcal{S}$ (from the German word Schlicht). For each function $F \in \mathcal{S}$, the function $H(z) = \sqrt[T]{F(z^T)} \in \mathcal{S}$ ($z \in \mathbb{D}$, $T = 1, 2, \dots$) and maps the unit disk to a domain with T-fold symmetry (T-fold symmetric function). G. Szegő suggested that for the coefficients of the T-symmetric univalent function

$$F(z) = z + \sum_{j=2}^{\infty} a_j z^{1+(j-1)T},$$

$\max |a_j| \leq O\left((1 + (j-1)T)^{-1+2/T}\right)$ [1, 2]. However, J. Littlewood showed [3] that this hypothesis is false for sufficiently large T. The results of N. Makarov [4] imply that the growth rate of coefficients of T-symmetric univalent functions equals $2/T - 1$ for $T \leq 2/B(1)$ and $B(1) - 1$ for $T \geq 2/B(1)$, where $B(t)$ is the universal integral means spectrum for the class of univalent functions. This theorem, together with the important paper by L. Carleson and P. Jones [5], suggest that G. Szegő's conjecture holds for $T \leq 8$ and fails for $T \geq 9$. D. Beliaev and S. Smirnov [6] proved that the conjecture is indeed wrong for $T \geq 9$. The general problem of finding the correct estimates is still wide open even for bounded functions, see, e.g., [7]. Thus, the transfer of the known extremal properties of general functions, belonging to the

class $\mathcal{S}$, to T-symmetric functions of the class S is not, generally, a trivial task.

Even more significant difficulties arise when trying to establish the well-known properties of univalent polynomials for T-symmetric univalent polynomials.

One of the most famous families of univalent polynomials solving various extremal problems is the T. Suffridge polynomials [8]:

$$S_{k,N}(z) = z + \sum_{j=2}^{N} \sigma_{j,k} z^j,$$

$$\sigma_{j,k} = \left(1 - \frac{j-1}{N}\right) \frac{\sin \frac{\pi k j}{N+1}}{\sin \frac{\pi k}{N+1}}, \; k = 1, 2, \ldots, N.$$

Note that Suffridge polynomials are the matter of greater part of the section on univalent polynomials in the well-known book on univalent functions [9].

It is easy to see that the coefficient $\sigma_{N,k} = \frac{(-1)^k}{N}$ possesses the extremal property, since the absolute value of the leading coefficient of a univalent polynomial, whose first coefficient is equal to one, can not exceed the value of $1/N$.

T. Suffridge proved that all the polynomials $S_{k,N}(z)$ are univalent in $\mathbb{D}$. Moreover, the zeros of the derivatives of polynomials $S_{k,N}(z)$ lie on the unit circle, and the image of the unit circle contains $N-1$ cusps [10]. Thus, the Suffridge polynomials are quasi-extremal polynomials in the sense of Ruscheweyh [11].

Also, T. Suffridge showed that the polynomial $S(z) := S_{1,N}(z)$ is extremal, in the sense that it maximizes the absolute value of each coefficient of any univalent polynomial of order N with the leading coefficient $1/N$.

In [12], one more extremal property of the polynomial $S(z)$ was established, namely there was solved the extremal problem of estimating the maximum value of the modulus of univalent polynomials with real coefficients of the form $P(z) = z + \sum_{j=2}^{N} a_j z^j$ in the disk

$\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$:

$$(2) \qquad \max_{P \in \mathcal{S}} \left\{ \max_{z \in \mathbb{D}} \{|P(z)|\} \right\} = S(1) = \frac{1}{4} \left(1 + \frac{1}{N} \right) \csc^2 \frac{\pi}{2N + 2}.$$

The corresponding extremal polynomial is unique up to alternation of signs, i.e., the polynomials $S(z)$ and $-S(-z)$ are extremal.

The following extremal property of the Suffridge polynomial is related to the problem of covering intervals and is achieved on the class of the univalent in $\mathbb{D}$ polynomials with real coefficients $P(z)$ with the normalization $P(1) = 1$. Namely,

$$(3) \qquad \min_P \{|P(-1)|\} = \frac{|S(-1)|}{S(1)} = \tan^2 \frac{\pi}{2(N + 1)}.$$

The polynomial $\frac{1}{S(1)} S(z)$ is again the only extremal polynomial in this problem [13].

Note that the Suffridge polynomials are used to approximate the Koebe function [13], some unexpected applications of extremal polynomials to the solution of control problems in nonlinear discrete systems are also found [14].

Obviously, the T-symmetric function $\sqrt[T]{S(z^T)}$ is not a polynomial. What polynomial with circlurar symmetry will have the properties analogous to those of the Suffridge polynomial? What values will the corresponding extremal quantities take in the considered above extremal problems? This paper is devoted to a discussion of these issues. Let us note only that even for $T = 2$, that is, in the case of odd polynomials, these problems are not easy.

The paper is organized as follows. In Section 2, polynomials with fold symmetry are presented as candidates for an analogue of the Suffridge polynomials. A number of hypotheses for the extremal properties of these polynomials, which are inherent in Suffridge polynomials, are put forward. The third section gives a description of the domain of univalence for trinomials with fold symmetry in the coefficient plane, using five parametric equations of the boundary. There are formulated constrained extremum problems for some functions of two variables, which are test ones for checking the stated hypotheses. The fourth section is auxiliary, two technical Lemmas are proved there. Section 5

contains the main Theorem of the paper, from which it follows that all the formulated hypotheses are true for trinomials with fold symmetry. The last section is devoted to a short discussion of the obtained results.

2. PROBLEM STATEMENT

In [15, 16], there was introduced a new class of T-symmetric polynomials of degree $N = 1 + (n-1)T$

$$(4) \quad S^{(T)}(z) = z + \sum_{j=2}^{n} \left(1 - \frac{(j-1)T}{1 + (n-1)T} \right) \prod_{k=1}^{j-1} \frac{\sin \frac{\pi(2 + T(k-1))}{2 + T(n-1)}}{\sin \frac{\pi T k}{2 + T(n-1)}} z^{T(j-1)+1}.$$

It is easy to check that $S^{(1)}(z) = S(z)$, $S^{(2)}(z) = -iS_{n,N}(iz)$, where $N = 2n - 1$, $i^2 = -1$. Hence, the polynomial $S^{(2)}(z)$ is univalent. Note that the leading coefficient of the polynomial $S^{(T)}(z)$ has the maximum possible value for univalent polynomials of degree $N = 1 + (n-1)T$ with the first coefficient equal to one: $\frac{1}{1+(n-1)T}$.

The works [16] [15] presented the results of some numerical experiments, which, together with the established results, made it possible to propose a number of hypotheses. Let us consider them.

Denote the class of univalent T-symmetric polynomials with real coefficients of degree $N = 1 + (n-1)T$ and with the first coefficient equal to one by $\mathcal{S}_N^T$. Hypotheses:

a) all the polynomials $S^{(T)}(z)$ are univalent in $\mathbb{D}$;

b) all the polynomials $S^{(T)}(z)$ are quasi-extremal in the sense of Ruscheweyh;

c) all the polynomials $S^{(T)}(z)$ are extremal, in the sense that they maximize the absolute value of each coefficient of any polynomial from $\mathcal{S}_N^T$ with the leading coefficient $1/N$;

d) $\max\limits_{P \in \mathcal{S}_N^T} \left\{ \max\limits_{z \in \overline{\mathbb{D}}} \{|P(z)|\} \right\} = S^{(T)}(1)$;

e) $\min\limits_{P \in \mathcal{S}_N^T} \left\{ \frac{1}{P(1)} \left| P\left(e^{i\frac{\pi}{T}}\right) \right| \right\} = \frac{1}{S^{(T)}(1)} \left| S^{(T)}\left(e^{i\frac{\pi}{T}}\right) \right|.$

Problem: check the truth of the formulated hypotheses for all $T = 1, 2, \ldots$ and $n = 3$. Note that in contrast with the cases $n = 1, 2$, the problem of determining the extremal properties of polynomials from different classes even for small $n \geq 3$ is usually far from trivial [17].

Let us only note that the problem $e)$ for $T = 2$ is solved in [14], where it is established that

$$\min_{P \in \mathcal{S}_N^2} \left\{ \frac{1}{P(1)} \left| P\left(e^{i\frac{\pi}{2}}\right) \right| \right\} = \frac{1}{S^{(2)}(1)} \left| S^{(2)}\left(e^{i\frac{\pi}{2}}\right) \right| = \left| \frac{S_{n,N}(-1)}{S_{n,N}(i)} \right| = \frac{1}{n},$$

and the extremal polynomial is unique.

3. Domain of univalence of T-symmetric trinomials in the coefficient space

In [18], there was considered the problem of constructing the domain of univalence for the trinomials

$$F(z) = z + az^k + bz^m$$

with complex coefficients. Let us present the result of this work for the case of real coefficients. So, let

$$U_{k,m} = \{(a,b) \in \mathbb{R}^2 : z + az^k + bz^m \text{ is univalent in } \mathbb{D}\}.$$

Let $\frac{k-1}{m-k} = \frac{p}{q}$ be the fraction after reduction. Define the following five curves in the plane:

$$\Gamma_1 = \left\{ (x,y) : x = \frac{\sin(m-1)t}{k\sin(m-k)t}, \; y = \frac{\sin(k-1)t}{n\sin(m-k)t}, \; t \in \mathbb{R} \right\},$$

$$\Gamma_2^s = \{(x,y) : kx + (-1)^{qs}my = (-1)^{ps+1}\}, \; s = 0,1,$$

$$\Gamma_3^s = \left\{ (x,y) : x = (-1)^{ps+1}\frac{\cos t}{\cos kt}\frac{\tan mt - m\tan t}{k\tan mt - m\tan kt}, \right.$$

$$\left. y = (-1)^{(p+q)s}\frac{\cos t}{\cos mt}\frac{\tan kt - k\tan t}{k\tan mt - m\tan kt}, \; t \in \mathbb{R} \right\}, \; s = 0,1.$$

It is noted further that these curves define the area $U_{k,m}$ in such a way that the boundary of this area is contained in the union of these five curves. The set $U_{k,m}$ is a component bounded by these curves, which contains zero.

A thorough analysis of the equations for the boundaries of the area $U_{k,m}$ shows that these equations can be simplified, and we also can identify the exact intervals of variation for the parameters in the parametric equations for setting the boundaries. Let us present these equations for the case $k = 1 + T$, $m = 1 + 2T$.

We have $\frac{k-1}{m-k} = 1$, $p = 1$, $q = 1$.

$$\frac{\cos t}{\cos (1+T)t} \frac{\tan (1+2T)t - (1+2T)\tan t}{(1+T)\tan (1+2T)t - (1+2T)\tan (1+T)t} =$$
$$= \frac{2T\sin (2+2T)t - (2+2T)\sin 2Tt}{T\sin (2+3T)t - (2+3T)\sin Tt},$$

$$\frac{\cos t}{\cos (1+2T)t} \frac{\tan (1+T)t - (1+T)\tan t}{(1+T)\tan (1+2T)t - (1+2T)\tan (1+T)t} =$$
$$= \frac{T\sin (2+T)t - (2+T)\sin Tt}{T\sin (2+3T)t - (2+3T)\sin Tt}.$$

Then the domain U_T of univalence of the trinomial $F(z) = z + az^{1+T} + bz^{1+2T}$ is bounded by the curves:

$$\Gamma_1 = \left\{ (x,y): \ x = t, \ y = \frac{1}{1+2T}, \right.$$
$$\left. t \in \left[-2\frac{1+T}{1+2T}\sin \frac{\pi}{2+2T}, 2\frac{1+T}{1+2T}\sin \frac{\pi}{2+2T} \right] \right\},$$

$$\Gamma_2^+ = \left\{ (x,y): \ x = t, \ y = \frac{(1+T)t-1}{1+2T}, \ t \in \left[0, \frac{4}{2+3T} \right] \right\},$$
$$\Gamma_2^- = \left\{ (x,y): \ x = -t, \ y = \frac{(1+T)t-1}{1+2T}, \ t \in \left[0, \frac{4}{2+3T} \right] \right\},$$
$$\Gamma_3^+ = \left\{ (x,y): \ x = A(t), \ y = B(t), \ t \in \left[0, \frac{\pi}{2+2T} \right] \right\},$$
$$\Gamma_3^- = \left\{ (x,y): \ x = -A(t), \ y = B(t), \ t \in \left[0, \frac{\pi}{2+2T} \right] \right\},$$

where

$$A(t) = \frac{2T\sin (2+2T)t - (2+2T)\sin 2Tt}{T\sin (2+3T)t - (2+3T)\sin Tt},$$
$$B(t) = \frac{T\sin (2+T)t - (2+T)\sin Tt}{T\sin (2+3T)t - (2+3T)\sin Tt}.$$

The boundaries of the intervals of variation for the parameters in defining the curves, which determine the boundary of the domain U_T, are computed as the roots of the corresponding equations. Note that the domain U_1 is defined in [19, 20], and optimization for U_1 was used in [21]. Therefore, the current work is a generalization of [21].

To demonstrate how the domain U_T contracts with the growth of T, let us picture this area for different T (Fig. 1). In Fig. 2, represent the boundaries Γ_1, Γ_2^+, Γ_2^-, Γ_3^+, Γ_3^- and the point that corresponds to the trinomial from family (4):

$$(5) \qquad S^{(T)}(z) = z + a^{(0)} z^{1+T} + b^{(0)} z^{1+2T},$$

where $a^{(0)} = 2 \frac{1+T}{1+2T} \sin \frac{\pi}{2+2T}$, $b^{(0)} = \frac{1}{1+2T}$.
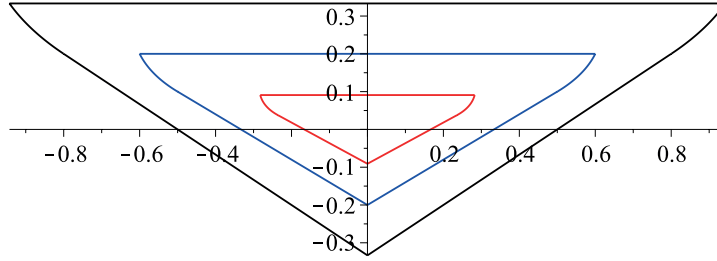

FIGURE 1. Domain U_T of univalence of the trinomial $F(z) = z + az^{1+T} + bz^{1+2T}$ in the parameter plane (a, b) for $T = 1$ (the boundary is shown in black), $T = 2$ (the boundary is shown in blue), $T = 5$ (the boundary is shown in red)
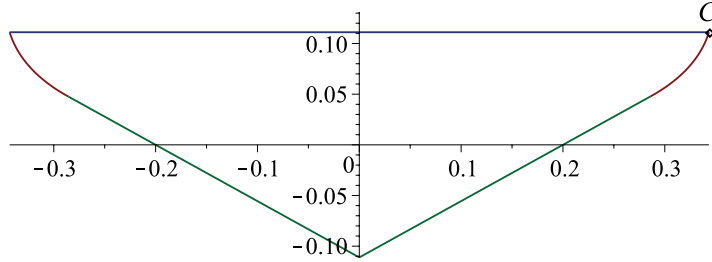

FIGURE 2. Curves Γ_1 (blue), Γ_2^+, Γ_2^- (green), Γ_3^+, Γ_3^- (red) and the point $C(a^{(0)}, b^{(0)})$ for $T = 4$

Note some properties of the domain U_T. This area, considered in the plane (a, b), has axial symmetry about the line Ob; the line Γ_1 defines polynomials with zeros of the derivative lying on the unit circle (quasi-extremal polynomials in the sense of Ruscheweyh). Thus, hypotheses *a)* and *b)* for trinomial (5) are confirmed.

To confirm the truth of hypotheses *c)*, *d)*, *e)*, we need to determine the extreme values of the functions

$$L_1(x, y) = x, \ L_2(x, y) = 1 + x + y, \ L_3(x, y) = \frac{1 - x + y}{1 + x + y}$$

in the domain U_T.

It is easy to show that these functions do not have stationary points in U_T (points at which both partial derivatives disappear) therefore we can restrict our attention to the boundary of the region. It will be shown that the functions $L_1(x, y)$ and $L_2(x, y)$ increase on the curve Γ_3^+, and the function $L_3(x, y)$ decreases on this curve. This will imply that

$$\max_{(x,y)\in U_T} \{L_1(x, y)\} = L_1(a^{(0)}, b^{(0)}) = a^{(0)} = 2\frac{1 + T}{1 + 2T} \sin \frac{\pi}{2 + 2T},$$

$$\max_{(x,y)\in U_T} \{L_2(x, y)\} = L_2(a^{(0)}, b^{(0)}) = 1 + a^{(0)} + b^{(0)} =$$

$$= \frac{1}{1 + 2T} \left(1 + 2(1 + T) \sin \frac{\pi}{2 + 2T} \right),$$

$$\min_{(x,y)\in U_T} \{L_3(x, y)\} = L_3(a^{(0)}, b^{(0)}) = \frac{1 - a^{(0)} + b^{(0)}}{1 + a^{(0)} + b^{(0)}} = \left(\frac{1 - \sin \frac{\pi}{2+2T}}{\cos \frac{\pi}{2+2T}} \right)^2.$$

Though the statement of the extremal problems is rather simple, we will need to overcome certain technical difficulties to solve them.

4. Auxiliary results

Lemma 1. *Let*

$$H(x, k) = -(1-x)^2 3^{2k} + (1+x)(3-2x)^{2k} - (1+x)(3-x) + (3-x)(1-2x)^{2k}.$$

Then, for $k \geq 3$ and $x \in (0, 1/2]$

$$(6) \qquad\qquad H(x, k) < 0,$$

and

$$(7) \qquad |H(x, k)| > \frac{1}{(2k + 1)(2k + 2)}|H(x, k + 1)|.$$

Proof. Let us start with the proof of (6). It is obvious that $(1+x)(3-x) > (3-x)(1-2x)^{2k}$, i.e. $-(1+x)(3-x) + (3-x)(1-2x)^{2k} < 0$. Further,

$$(1-x)^2 3^{2k} - (1+x)(3-2x)^{2k} = 3^{2k}\left((1-x)^2 - (1+x)(1-2x/3)^{2k}\right).$$

For $k \geq 3$, $x \in (0, 1/2]$, the following inequality holds:

$$(1-x)^2 - (1+x)(1-2x/3)^{2k} > (1-x)^2 - (1+x)(1-2x/3)^6.$$

The polynomial $(1-x)^2 - (1+x)(1-2x/3)^6$ has two complex conjugate zeros and five real ones: $x_1 = -0.71\ldots$, $x_2 = 0$, $x_3 = 0.91\ldots$, $x_4 = 1.04\ldots$, $x_5 = 3$, moreover, this polynomial is positive for $x \in (0, 0.91)$, hence, for $x \in (0, 1/2]$ as well. Therefore, $-(1-x)^2 3^{2k} + (1+x)(3-2x)^{2k} < 0$. The sum of two negative polynomials is a negative polynomial. Thus, (6) is proven.

Proof of inequality (7) follows the same ideas. Let us prove a slightly stronger inequality

$$|H(x,k)| > \frac{1}{3k^2}|H(x,k+1)|.$$

Let us change the inequality

$$(1+x)(3-x)-(3-x)(1-2x)^{2k} > \frac{1}{3k^2}\left((1+x)(3-x) - (3-x)(1-2x)^{2k}(1-2x)^2\right)$$

to

$$(1+x) > (1-2x)^{2k}\frac{3k^2 - (1-2x)^2}{3k^2 - 1}.$$

The above inequality is obvious because the left hand side is an increasing function while the right hand side is decreasing.

Now, let us show that

$$(1-x)^2 3^{2k}-(1+x)(3-2x)^{2k} > \frac{1}{3k^2}\left((1-x)^2 3^{2k}9 - (1+x)(3-2x)^{2k}(3-2x)^2\right).$$

To do that let us write the inequality in the form

$$(1-x)^2 > (1+x)\left(1-\frac{2x}{3}\right)^{2k}\frac{k^2 - 3\left(1-\frac{2x}{3}\right)^2}{k^2 - 3}.$$

Now, take into account that for $k \geq 3$ and $x \in (0, 1/2]$ we have

$$\frac{k^2 - 3\left(1-\frac{2x}{3}\right)^2}{k^2 - 3} < 1 + \frac{4}{k^2 - 3}x \leq 1 + \frac{2x}{3}$$

and

$$\left(1 - \frac{2x}{3}\right)^{2k} \le \left(1 - \frac{2x}{3}\right)^{6}.$$

Note that the polynomial $(1 - x)^2 - (1 + x)(1 - 2x/3)^6(1 + 2x/3)$ has two conjugate complex roots and six reals

$$x_1 = -1.67..., x_2 = -0.27..., x_3 = 0, x_4 = 0.87..., x_5 = 1.05..., x_5 = 2.71...$$

Moreover, this polynomial is positive for $x \in (0, 0.87)$, hence it is positive for $x \in (0, 1/2]$. Thus, the inequality (7) is valid.

The lemma is proved. $\qquad\square$

Lemma 2. *Let*

$$G(\tau, \alpha) = (1 + \alpha) \cos (3 - 2\alpha)\tau + (3 - \alpha) \cos (1 - 2\alpha)\tau -$$
$$-(1 - \alpha)^2 \cos 3\tau - (1 + \alpha)(3 - \alpha) \cos \tau.$$

Then $G(\tau, \alpha) > 0$ when $\tau \in (0, \pi/2)$, $\alpha \in (0, 1/2]$.

Proof. Using the cosine series expansion formula, we can write

$$(8) \qquad G(\tau, \alpha) = \sum_{k=0}^{\infty}(-1)^k \frac{1}{(2k)!} H(\alpha, k)\tau^{2k}.$$

Note that $H(\alpha, 0) = H(\alpha, 1) = H(\alpha, 2) \equiv 0$. By (7), the series (8) is an alternating series, and $H(\alpha, 3) < 0$. Using Leibniz's Theorem on estimating a function with the partial sums of its alternating series expansion, we obtain the estimate

$$G(\tau, \alpha) > \tau^6 \left(|H(\alpha, 3)| - |H(\alpha, 4)|\tau^2\right) = |H(\alpha, 4)|\tau^6 \left(\frac{14}{5 - 4\alpha + 2\alpha^2} - \tau^2\right).$$

When $\alpha \in (0, 1/2]$, $\dfrac{14}{5 - 4\alpha + 2\alpha^2} > \dfrac{14}{5}$. Then

$$G(\tau, \alpha) > |H(\alpha, 4)|\tau^6 \left(\frac{14}{5} - \left(\frac{\pi}{2}\right)^2\right) > 0$$

for $\tau \in (0, \pi/2)$. The lemma is proved.

$\qquad\square$

5. Main results

Let us find the extreme values of the functions

$$L_1(x, y) = x, \ L_2(x, y) = 1 + x + y, \ L_3(x, y) = \frac{1 - x + y}{1 + x + y}$$

in the domain U_T.

Theorem 1. *The functions $L_1(x, y)$ and $L_2(x, y)$ attain the maximum over the region U_T at the upper right corner, while the function $L_3(x, y)$ attains its minimum there.*

It follows from the Theorem that hypotheses *c), d), e)* are true.

Indeed, the point (x, y) in the domain corresponds to a univalent polynomial $z + xz^{1+T} + yz^{1+2T}$. In particular, among univalent polyno- mials $z + xz^{1+T} + \frac{1}{1 + 2T} z^{1+2T}$ the polynomial $S^{(T)}(z)$ has the largest coefficients, the corresponding point (x, y) is at the upper right corner of the region. This proves conjecture c).

Further, due to symmetry of the region and situation in an upper half-plane we might assume both x and y to be positive in consideration of conjecture d). In that case the maximum of the polynomial attains at $z = 1$, i.e.

$$\max_{z \in \mathbb{D}} \left\{ z + xz^{1+T} + yz^{1+2T} \right\} = 1 + x + y.$$

The maximum of $1 + x + y$ attains at the upper right corner of the domain, hence the value $|S^{(T)}(1)|$ is the largest one, which proves con- jecture d).

Conjecture e) follows by similar argument.

The extremal polynomial that corresponds to hypothesis *e)* is unique: $S^{(T)}(z)$; in the problems corresponding to hypotheses *c)* and *d)*, these are the polynomials $S^{(T)}(z), -S^{(T)}(-z)$.

Now, let us prove the theorem.

Proof. Because functions $L_j(x, y)$ do not have stationary points in U_T and due to the symmetry of the domain U_T, we can restrict ourselves to the boundary and to the case $x \geq 0$. Obviously, the functions

$L_1(x, y)$ and $L_2(x, y)$ increase on the curves Γ_1 and Γ_2^+. And the function $L_3(x, y)$ decreases on these curves. It remains to consider the behavior of these functions on the curve Γ_3^+.

Since $\frac{1-x+y}{1+x+y} = 1 - \frac{2}{1+\frac{1+y}{x}}$, instead of the initial functions $L_2(x, y)$, $L_3(x, y)$ it is more convenient to consider the functions $\hat{L}_2(x, y) = y$, $\hat{L}_3(x, y) = \frac{1+y}{x}$.

We represent the parametric equations defining the curve Γ_3^+ in the form

$$\Gamma_3^+ = \left\{ (x, y) : x = \frac{U(t)}{W(t)}, \ y = \frac{V(t)}{W(t)}, \ t \in \left[0, \frac{\pi}{2 + 2T}\right] \right\},$$

where

$$U(t) = 2T \sin (2 + 2T)t - (2 + 2T) \sin 2Tt,$$
$$V(t) = T \sin (2 + T)t - (2 + T) \sin Tt,$$
$$W(t) = T \sin (2 + 3T)t - (2 + 3T) \sin Tt.$$

Make the substitution $\tau = (1 + T)t$, $\alpha = \frac{1}{1+T}$. Then $\tau \in [0, \pi/2]$, $\alpha \in (0, 1/2]$. The functions $U(t)$, $V(t)$, $W(t)$ transform into the functions

$$\hat{U}(\tau) = \frac{1}{\alpha}(2 \sin (2(1 - \alpha))\tau - 2(1 - \alpha) \sin 2\tau),$$

$$\hat{V}(\tau) = \frac{1}{\alpha}((1 + \alpha) \sin (1 - \alpha)\tau - (1 - \alpha) \sin (1 + \alpha)\tau),$$

$$\hat{W}(\tau) = \frac{1}{\alpha}((3 - \alpha) \sin (1 - \alpha)\tau - (1 - \alpha) \sin (3 - \alpha)\tau),$$

and

$$\Gamma_3^+ = \left\{ (x, y) : x = \frac{\hat{U}(\tau)}{\hat{W}(\tau)}, \ y = \frac{\hat{V}(\tau)}{\hat{W}(\tau)}, \ \tau \in \left[0, \frac{\pi}{2}\right] \right\}.$$

Find the derivatives

$$\hat{U}'(\tau) = \frac{8}{\alpha}(1 - \alpha) \sin \alpha\tau \sin (2 - \alpha)\tau,$$

$$\hat{V}'(\tau) = \frac{2}{\alpha}(1 - \alpha)(1 + \alpha) \sin \alpha\tau \sin \tau,$$

$$\hat{W}'(\tau) = \frac{2}{\alpha}(3 - \alpha)(1 - \alpha) \sin (2 - \alpha)\tau \sin \tau.$$

Then

$$\left(\frac{\hat{U}(\tau)}{\hat{W}(\tau)}\right)' = \frac{2(1-\alpha)\sin{(2-\alpha)\tau}}{\left(\hat{W}(\tau)\right)^2}G(\tau,\alpha),$$

$$\left(\frac{\hat{V}(\tau)}{\hat{W}(\tau)}\right)' = \frac{2(1-\alpha)\sin{\tau}}{\left(\hat{W}(\tau)\right)^2}G(\tau,\alpha),$$

$$\left(\frac{\hat{V}(\tau)}{\hat{U}(\tau)}\right)' = \frac{2(1-\alpha)\sin{\alpha\tau}}{\left(\hat{U}(\tau)\right)^2}G(\tau,\alpha),$$

$$\left(\frac{\hat{W}(\tau)}{\hat{U}(\tau)}\right)' = -\frac{2(1-\alpha)\sin{(2-\alpha)\tau}}{\left(\hat{U}(\tau)\right)^2}G(\tau,\alpha),$$

and

$$\left(\frac{\hat{V}(\tau)+\hat{W}(\tau)}{\hat{U}(\tau)}\right)' = -\frac{4}{\left(\hat{U}(\tau)\right)^2}(1-\alpha)\cos{\tau}\sin{(1-\alpha)\tau}G(\tau,\alpha).$$

By Lemma 2, $G(\tau,\alpha) > 0$ when $\tau \in (0,\pi/2)$, $\alpha \in (0,1/2]$. Therefore,

$$\left(\frac{\hat{U}(\tau)}{\hat{W}(\tau)}\right)' > 0, \quad \left(\frac{\hat{V}(\tau)}{\hat{W}(\tau)}\right)' > 0, \quad \left(\frac{\hat{V}(\tau)+\hat{W}(\tau)}{\hat{U}(\tau)}\right)' < 0.$$

Hence, we proved that on the curve Γ_3^+:

1) the function $L_1(x,y) = x$ increases;
2) the function $\hat{L}_2(x,y) = y$ increases;
3) the function $\hat{L}_3(x,y) = \frac{1+y}{x}$ decreases.

The theorem is proved. $\qquad\square$

6. Discussion of the results

Of course, the next problem in the queue for consideration is the case of polynomials of degree $n = 4$, or more generally, $n \geq 4$. The domains of univalence in these cases are not known even for $T = 1$. This does not prevent considering hypotheses *a)-e)* in the general situation, but greatly complicates the solution.

The next natural question is the question about the maximum contraction of the unit disk in the direction of the real axis for univalent

polynomials with the schlicht normalization $F(0) = 0$, $F'(0) = 1$. Note that in this case the Suffridge polynomials are not optimal [21, 22]. Thus, changing the normalization $F(0) = 0$, $F'(0) = 1$ to $F(0) = 0$, $F(1) = 1$ leads not only to a change of the extreme value, but also to a change of the extremal polynomial, which is quite interesting.

In [15], there were introduced the polynomials $s^{(T)}(z)$ that formally differ from the polynomials $S^{(T)}(z)$ defined by (4). The following asymptotics was established:

$$\frac{1}{s^{(T)}(1)} \left| s^{(T)} \left(e^{i\frac{\pi}{T}} \right) \right| \sim c_T n^{-2/T}$$

as $n \to \infty$, where $c_T = \pi^{2/T-1}\Gamma^2(1/T + 1/2)$, $\Gamma(\cdot)$ is the gamma function.

However, already [16] introduces and proves, for special cases, the conjecture that the polynomials $s^{(T)}(z)$ coincide with the polynomials $S^{(T)}(z)$. In the same work, the following relation was shown:

$$\max_{z \in \overline{\mathbb{D}}} \left\{ |s^{(T)}(z)| \right\} = s^{(T)}(1) \sim \frac{1}{c_T 2^{2/T}} n^{2/T}$$

as $n \to \infty$, which is a quantitative refinement of hypothesis d).

7. Acknowledgment

References

[1] J. Basilevitsh. Zum Koeffizientenproblem der schlichten Funktionen. *Math. Z.*, 43(2):211–228, 1936.

[2] W.I. Lewin. Ein beitrag zum koeffizientenproblem der schlichten funktionen. *Math. Z.*, 38:306–311, 1934.

[3] J.E. Littlewood. On the coefficients of schlicht functions. *Q. J. Math.*, 9(33):14–20, 1938.

[4] N.G. Makarov. Fine structure of harmonic measure. *St. Petersburg Math. J.*, 10(2):217–268, 1999.

[5] L. Carleson, P.W. Jones. On coefficient problems for univalent functions and conformal dimension. *Duke Math. J.*, 66:169–206, 1992.

[6] D. Beliaev, S. Smirnov. Harmonic measure on fractal sets. In *European Congress of Mathematics*, pages 41–59. Eur. Math. Soc., Zürich, 2005.

[7] D. Beliaev, S. Smirnov. Random conformal snowflakes. *Ann. Math.*, 172(1):597–615, 2010.

[8] T. Suffridge. On univalent polynomials. *J. London Math. Soc.*, 44:496–504, 1969.

[9] P.L. Duren. *Univalent Functions.* Grundlehren der mathematischen Wissenschaften, 1983.

[10] A. Cordova, St. Ruscheweyh. On maximal ranges of polynomial spaces in the unit disk. *Constr. Approx.*, 5:309–328, 1989.

[11] C. Genthner, St. Ruscheweyh, L. Salinas. A criterion for quasi-simple plane curves. *Comput. Methods Funct. Theory*, 2(1):281–291, 2003.

[12] M. Brandt. Representation formulas for the class of typically real polynomials. *Math. Nachr.*, 144:29–37, 1989.

[13] D. Dimitrov. Extremal univalent polynomials subordinating the Koebe function. *East J. Approx.*, 11(1):47–56, 2005.

[14] D. Dmitrishin, P. Hagelstein, A. Khamitova, A. Korenovskyi, A. Stokolos . Fejér polynomials and control of nonlinear discrete systems. *Constr. Approx.*, 51(2):383–412, 2020.

[15] D. Dmitrishin, A. Stokolos, M. Tohaneanu. Searching for cycles in non-linear autonomous discrete dynamical systems. *N.Y. J. Math.*, 25:603–626, 2019.

[16] D. Dmitrishin, A. Smorodin, A. Stokolos, M. Tohaneanu. Symmetrization of Suffridge polynomials and approximation of T-symmetric Koebe functions. *J. Math. Anal. Appl.*, 503(2), 2020.

[17] S. Ignaciuk, M. Parol. On the Koebe Quarter Theorem for certain polynomials. *Anal. Math. Phys.*, 11(2):67–79, 2021.

[18] G. Schmieder. Univalence and zeros of complex polynomials. In *Handbook of complex analysis geometric function theory*, volume 2, pages 339–350. Martin-Luther-Universität Halle-Wittenberg, Halle, Germany, Elsevier, 2005.

[19] D.A. Brannan. Coefficient regions for univalent polynomials of small degree. *Mathematika*, 14:165–169, 1967.

[20] V.F. Cowling, W.C. Royster. Domains of variability for univalent polynomials. *Proc. Am. Math. Soc.*, 19:767–772, 1968.

[21] K. Dyakonov D. Dmitrishin and A. Stokolos. Univalent polynomials and koebe's one-quarter theorem. *Anal. Math. Phys.*, pages 1–14, 2019.

[22] M. Brandt. Variationsmethoden für in der einheitskreisscheibe schlichte polynome. *PhD thesis. Humboldt-Univ. Berlin*, 1987.

Department of Applied Mathematics, Odessa National Polytechnic University, Odessa, 65044, Ukraine

Email address: dmitrishin@op.edu.ua

Department of Mathematics, Georgia Southern University, Statesboro GA, 30460, USA

Email address: dagray@georgiasouthern.edu

Department of Mathematics, Georgia Southern University, Statesboro GA, 30460, USA

Email address: astokolos@georgiasouthern.edu